

**The two-fermion relativistic wave equations
of Constraint Theory in the Pauli-Schrödinger form**

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Abstract

The two-fermion relativistic wave equations of Constraint Theory are reduced, after expressing the components of the 4×4 matrix wave function in terms of one of the 2×2 components, to a single equation of the Pauli-Schrödinger type, valid for all sectors of quantum numbers. The potentials that are present belong to the general classes of scalar, pseudoscalar and vector interactions and are calculable in perturbation theory from Feynman diagrams. In the limit when one of the masses becomes infinite, the equation reduces to the two-component form of the one-particle Dirac equation with external static potentials. The Hamiltonian, to order $1/c^2$, reproduces most of the known theoretical results obtained by other methods. The gauge invariance of the wave equation is checked, to that order, in the case of QED. The role of the c.m. energy dependence of the relativistic interquark confining potential is emphasized and the structure of the Hamiltonian, to order $1/c^2$, corresponding to confining scalar potentials, is displayed.

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I. INTRODUCTION

The use of the manifestly covariant formalism with constraints [1] in the two-body problem [2,3] leads to a Poincaré invariant description of the dynamics of the system with the correct number of degrees of freedom. Furthermore, the potentials that appear in the corresponding wave equations are calculable, in perturbation theory, in terms of the kernel of the Bethe-Salpeter equation, and therefore allow one to deal with quantum field theoretic problems.

For two spin-0 particles, the system is described by two independent wave equations, which are generalizations of the individual Klein-Gordon equation of each particle, including now the mutual interaction potential. The compatibility condition of the two equations imposes certain restrictions on the structure of the potential and leads, in a covariant form, to an elimination of the relative energy variable. One then ends up with a final, manifestly covariant, three-dimensional eigenvalue equation that describes the relative motion of the two particles. This equation is very similar in form, except for kinematic energy dependent factors, to the Schrödinger (or Klein-Gordon) equation : it is a second order differential equation in the three spacelike coordinates (for local potentials) and therefore the usual techniques of nonrelativistic quantum mechanics are applicable to it.

For two spin-1/2 particles, the system is described by two independent Dirac type equations. Again, the compatibility condition imposes restrictions on the structure of the potentials and eliminates the relative energy variable; but, because of the presence of the Dirac matrices, the reduction to a final eigenvalue equation is not straightforward. Except for certain classes of interaction, as the pseudoscalar interaction and the vector interaction without temporal components, the reduction process is rather complicated and dependent on the way of eliminating the components of the spinor wave function in terms of one of them. Up to now, no single Pauli-Schrödinger type equation was obtained from this procedure, capable of describing general cases of interaction and all sectors of quantum numbers, although it was possible to obtain two coupled Pauli-Schrödinger type equations for the sectors of quantum

numbers with $\ell = |j \pm 1|$ and single equations for the sectors with $\ell = j$ [4].

The purpose of the present paper is to show that it is possible, for general combinations of scalar, pseudoscalar and vector interactions, the latter being considered in arbitrary covariant gauges, to reduce the wave equations describing two spin-1/2 particle systems to a single Pauli-Schrödinger type equation, valid for all sectors of quantum numbers. This is achieved by decomposing the 4×4 spinor wave function (of a fermion-antifermion system, say) along 2×2 components defined, in the c.m. frame, by the basis of the matrices 1 , γ_0 , γ_5 and $\gamma_0\gamma_5$, rather than the usual basis $(1 \pm \gamma_0)$ and $(1 \pm \gamma_0)\gamma_5$.

The above reduction is obtained in the case of quasilocal approximations of the interaction potentials, i.e., for potentials that are functions of the relative coordinates and, eventually, of the c.m. energy. This does not cover the more general case of integral operators (in the three spacelike coordinates), but it is generally admitted that a local expression of the potential may provide a valid zeroth order approximation of the total interaction, from which one may develop perturbative calculations for the nonlocal effects. Furthermore, an appropriate c.m. energy dependence of the potential takes into account the leading contribution of the nonlocal effects.

The final Pauli-Schrödinger type equation that is obtained provides explicit and simple means of controlling the relativistic effects of each type of interaction.

The contents of the paper are summarized as follows. In Sec. II, we review the main aspects of the relativistic wave equations of Constraint Theory for a fermion-antifermion system. The calculation of the potentials from one-particle exchange Feynman diagrams is presented in Appendix A.

In Sec. III, we reduce these wave equations to a single two-body Pauli-Schrödinger type equation, for the case of a general combination of scalar, pseudoscalar and vector interactions. Some technical details of the reduction process and useful formulas for the γ -matrices and the spin and orbital angular momentum operators are presented in Appendix B.

In Sec. IV, we study the limit of the above equation when one of the particles becomes

infinitely massive. It reduces to the Pauli-Schrödinger form of the one-particle Dirac equation in the presence of the static potential created by the heavy particle. In Appendix C, we show that, for the class of ladder and crossed ladder type diagrams, it is only the one-particle exchange diagram which contributes in this limit to the expression of the potential.

Section V deals with the nonrelativistic limit, to order $1/c^2$, of the above two-body Pauli-Schrödinger type equation. Most of the known theoretical results obtained by other methods are reproduced. In the case of QED, its gauge invariance, to order $1/c^2$, is checked by considering the photon propagator in arbitrary covariant gauges. The role of the c.m. energy dependence of the relativistic potentials is emphasized and the structure of the Hamiltonian, for confining scalar potentials, ready for comparisons with quantum field theoretic results, is displayed.

Summary and concluding remarks follow in Sec. VI.

II. THE TWO-BODY RELATIVISTIC WAVE EQUATIONS OF CONSTRAINT THEORY

For a system of two spin-1/2 particles, composed of one fermion of mass m_1 and one antifermion of mass m_2 , say, Constraint Theory imposes two independent wave equations, which are generalizations of the Dirac equation satisfied by each particle in the free case. These wave equations have not a unique form and can be modified by wave function transformations, but a convenient form, where several properties can be easily read off, is the following one [3] :

$$(\gamma_1 \cdot p_1 - m_1) \tilde{\Psi} = (-\gamma_2 \cdot p_2 + m_2) \tilde{V} \tilde{\Psi} , \quad (2.1a)$$

$$(-\gamma_2 \cdot p_2 - m_2) \tilde{\Psi} = (\gamma_1 \cdot p_1 + m_1) \tilde{V} \tilde{\Psi} . \quad (2.1b)$$

Here, $\tilde{\Psi}$ is a sixteen-component spinor wave function of rank two and is represented as a 4×4 matrix :

$$\tilde{\Psi} = \tilde{\Psi}_{\alpha_1, \alpha_2}(x_1, x_2) \quad (\alpha_1, \alpha_2 = 1, \dots, 4) , \quad (2.2)$$

where $\alpha_1(\alpha_2)$ refers to the spinor index of particle 1(2). γ_1 is the Dirac matrix γ acting in the subspace of the spinor of particle 1 (index α_1); it acts on $\tilde{\Psi}$ from the left. γ_2 is the Dirac matrix acting in the subspace of the spinor of particle 2 (index α_2); it acts on $\tilde{\Psi}$ from the right; this is also the case of products of γ_2 matrices, which act on $\tilde{\Psi}$ from the right in the reverse order :

$$\begin{aligned} \gamma_{1\mu} \tilde{\Psi} &\equiv (\gamma_\mu)_{\alpha_1 \beta_1} \tilde{\Psi}_{\beta_1 \alpha_2} , & \gamma_{2\mu} \tilde{\Psi} &\equiv \tilde{\Psi}_{\alpha_1 \beta_2} (\gamma_\mu)_{\beta_2 \alpha_2} , \\ \gamma_{2\mu} \gamma_{2\nu} \tilde{\Psi} &\equiv \tilde{\Psi}_{\alpha_1 \beta_2} (\gamma_\nu \gamma_\mu)_{\beta_2 \alpha_2} , & \sigma_{a\alpha\beta} &= \frac{1}{2i} [\gamma_{a\alpha}, \gamma_{a\beta}] \quad (a = 1, 2) . \end{aligned} \quad (2.3)$$

In Eqs. (2.1) p_1 and p_2 represent the momentum operators of particles 1 and 2, respectively. \tilde{V} is a Poincaré invariant potential.

Equations (2.1) must be compatible among themselves. This is enforced in two steps. First, one multiplies Eq. (2.1a) by $(\gamma_1 \cdot p_1 + m_1)$ and Eq. (2.1b) by $(-\gamma_2 \cdot p_2 + m_2)$ and subtracts the two equations from each other. This yields the constraint :

$$[(p_1^2 - p_2^2) - (m_1^2 - m_2^2)] \tilde{\Psi} = 0 , \quad (2.4)$$

which allows one to eliminate the relative energy variable in a covariant form. For eigenfunctions of the total momentum operator P , the solution of Eq. (2.4) is :

$$\tilde{\Psi} = e^{-iP.X} e^{-i(m_1^2 - m_2^2)P.x/(2P^2)} \tilde{\psi}(x^T) , \quad (2.5)$$

where we have used notations from the following definitions :

$$\begin{aligned} P &= p_1 + p_2 , \quad p = \frac{1}{2}(p_1 - p_2) , \quad M = m_1 + m_2 , \\ X &= \frac{1}{2}(x_1 + x_2) , \quad x = x_1 - x_2 , \quad \mu = \frac{m_1 m_2}{M} . \end{aligned} \quad (2.6)$$

We also define transverse and longitudinal components of four-vectors with respect to the total momentum P :

$$\begin{aligned} q_\mu^T &= q_\mu - \frac{(q.P)}{P^2} P_\mu , \quad q_\mu^L = (q.\hat{P}) \hat{P}_\mu , \quad \hat{P}_\mu = P_\mu / \sqrt{P^2} , \\ q_L &= q.\hat{P} , \quad P_L = \sqrt{P^2} . \end{aligned} \quad (2.7)$$

This decomposition is manifestly covariant. In the c.m. frame the transverse components reduce to the three spacelike components, while the longitudinal component reduces to the timelike component of the corresponding four-vector. (Note that $x^{T2} = -\mathbf{x}^2$ in the c.m. frame.)

Second, one multiplies Eq. (2.1a) by $(-\gamma_2.p_2 - m_2)$ and Eq. (2.1b) by $(\gamma_1.p_1 - m_1)$ and subtracts the two equations from each other. Taking into account the constraint (2.4), one ends up with the new constraint :

$$[p_1^2 - p_2^2 , \tilde{V}] \tilde{\Psi} = 0 , \quad (2.8)$$

which concerns the potential \tilde{V} . By noticing that $p_1^2 - p_2^2 = 2P_L p_L$, this equation is satisfied by demanding that \tilde{V} , which is Poincaré invariant, be independent of the longitudinal relative coordinate x_L :

$$\tilde{V} = \tilde{V}(x^T, P_L, p^T, \gamma_1, \gamma_2) . \quad (2.9)$$

No other constraints are found. Equations (2.5) and (2.9) show that the internal dynamics of the system is three-dimensional, besides the spin degrees of freedom, described by the three-dimensional transverse coordinate x^T .

To find the connection of Eqs. (2.1) with the Bethe-Salpeter equation [5,6], one projects the latter on the constraint hypersurface (2.4) by expanding the right-hand side of the equation on that hypersurface [7]. One ends up precisely with Eqs. (2.1), where now the potential \tilde{V} is calculable, in perturbation theory, in terms of Feynman diagrams.

The relationship between the potential \tilde{V} and Feynman diagrams can be summarized by the following Lippmann-Schwinger-Quasipotential type [8–15] equation (momentum integrations, as well as the total momentum P are not explicitly written) :

$$\begin{aligned} \tilde{V} - \tilde{T} - \tilde{V}G_0\tilde{T} &= 0 , \\ \tilde{T}(p^T, p'^T) &\equiv \frac{i}{2P_L} \left[T(p, p') \right]_{C(p), C(p')} , \end{aligned} \quad (2.10)$$

where :

- i) T is the off-mass shell fermion-antifermion scattering amplitude (amputated four-point connected Green's function);
- ii) C is the constraint (2.4) :

$$C(p) \equiv (p_1^2 - p_2^2) - (m_1^2 - m_2^2) = 2P_L p_L - (m_1^2 - m_2^2) \approx 0 ; \quad (2.11)$$

in Eq. (2.10) the external momenta of the amplitude T are submitted to the constraint C ;

- iii) G_0 is defined as :

$$G_0(p_1, p_2) = \tilde{S}_1(p_1) \tilde{S}_2(-p_2) H_0 , \quad (2.12)$$

where \tilde{S}_1 and \tilde{S}_2 are the propagators of the two fermions, respectively, in the presence of the constraint (2.11), and H_0 is the Klein-Gordon operator, also in the presence of the constraint (2.11) :

$$H_0 = (p_1^2 - m_1^2) \Big|_C = (p_2^2 - m_2^2) \Big|_C = \frac{P^2}{4} - \frac{1}{2}(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4P^2} + p^{T^2} . \quad (2.13)$$

The series expansion of Eq. (2.10) involves not only the usual Feynman diagrams of the amplitude T , but also additional diagrams, which we call “constraint diagrams”, coming from the iteration of the $\tilde{V}G_0$ term; they are obtained from the reducible diagrams by the replacement in a box diagram of the fermion and antifermion propagators by the factor G_0 (2.12) together with the δ -function of the constraint (2.11). It is the presence of these diagrams, as well as of the c.m. energy factor in the function $\delta(C)$, that cures the Bethe-Salpeter equation of some of its deficiencies [7]. (See also Appendix C.) [We have included the c.m. energy factor $1/(2P_L)$ in the definition of \tilde{T} , and therefore, by imposing on G_0 the constraint (2.11), the integrations relative to G_0 in Eq. (2.10) may be considered as three-dimensional.]

In general \tilde{V} is an integral operator in x^T . However, in one-particle exchange diagrams \tilde{V} turns out to be a function of x^T and P_L , with definite dependences on the γ -matrices. It would also be meaningful to approximate, as a zeroth order approximation, multiparticle exchange contributions by appropriate local functions and thus to use for \tilde{V} quasilocal expressions (in x^T and P_L). Furthermore, in perturbation theory, the c.m. energy (P_L) dependence of the potential takes into account the leading contribution of the nonlocal effects. Therefore, we shall confine ourselves in the present paper to quasilocal expressions of \tilde{V} . The explicit expressions of \tilde{V} in lowest order of perturbation theory, for the scalar, pseudoscalar and vector interactions, are presented in Appendix A.

The scalar product associated with the solutions of Eqs. (2.1) can be constructed by means of tensor currents of rank two, satisfying two independent conservation laws, with respect to x_1 and x_2 [3]. Another way of proceeding consists in using the integral equation of the Green’s function [12,15]. From either method, one finds for the norm of $\tilde{\psi}$ [Eq. (2.5)] the formula (in the c.m. frame) :

$$\int d^3\mathbf{x} \, Tr \left\{ \tilde{\psi}^\dagger \left[1 - \tilde{V}^\dagger \tilde{V} + 4\gamma_{10}\gamma_{20}P_0^2 \frac{\partial \tilde{V}}{\partial P^2} \right] \tilde{\psi} \right\} = 2P_0 . \quad (2.14)$$

(The potentials \tilde{V} that are considered in the sequel commute with the matrix product $\gamma_{1L}\gamma_{2L}$.)

This formula shows that for energy independent potentials (in the c.m. frame) the norm of $\tilde{\psi}$ is not positive definite. In order to reach such a situation, it is sufficient that the potential \tilde{V} satisfy the inequality

$$\frac{1}{4}Tr(\tilde{V}^\dagger \tilde{V}) < 1 , \quad (2.15)$$

which, in turn imposes restrictions on the high order (multiparticle exchange) contributions to \tilde{V} . Once condition (2.15) is satisfied, one is allowed to make the wave function transformation

$$\tilde{\Psi} = [1 - \tilde{V}^\dagger \tilde{V}]^{-\frac{1}{2}} \Psi \quad (2.16)$$

and to reach a representation of the wave equations (2.1), where the norm for energy independent potentials is the free norm.

In this connection, it was observed by Crater and Van Alstine [16] that the parametrization

$$\tilde{V} = \tanh V \quad (2.17)$$

satisfies condition (2.15) and also has the property that in the new representation (2.17) one obtains familiar Dirac type wave equations, where each particle appears as placed in the external potential created by the other particle, the latter potential having the same tensor nature as potential V of Eq. (2.17).

We shall henceforth adopt parametrization (2.17) for potential \tilde{V} and, according to Eq. (2.16), shall introduce the wave function transformation :

$$\tilde{\Psi} = (\cosh V) \Psi , \quad (2.18)$$

V being assumed to be hermitian. The norm of the new wave function Ψ then becomes (in the c.m. frame) :

$$\int d^3\mathbf{x} \, Tr \left\{ \psi^\dagger \left[1 + 4\gamma_{10}\gamma_{20}P_0^2 \frac{\partial V}{\partial P^2} \right] \psi \right\} = 2P_0 . \quad (2.19)$$

(The relationship between Ψ and ψ is the same as in Eq. (2.5).)

The new potential V will be chosen in the major part of this paper as a general combination of scalar, pseudoscalar and vector potentials :

$$V = V_1 + \gamma_{15}\gamma_{25}V_3 + \gamma_1^\mu\gamma_2^\nu (g_{\mu\nu}^{LL}V_2 + g_{\mu\nu}^{TT}U_4 + \frac{x_\mu^T x_\nu^T}{x^{T2}}T_4) , \quad (2.20)$$

where the potentials V_1 , V_2 , etc., are functions of x^{T2} and P_L . (The index notations of the potentials will be clarified in Sec. III.) The dependence of the vector potentials on the gauge chosen for the photon propagator in lowest order of perturbation theory is displayed in Appendix A. Also notice that the difference between potentials \tilde{V} and V shows up starting from third order diagrams.

When parametrization (2.17) is adopted and the change of function (2.18) is used, Eqs. (2.1) take the form :

$$(\gamma_1 \cdot p_1 - m_1) \cosh V \psi = (-\gamma_2 \cdot p_2 + m_2) \sinh V \psi , \quad (2.21a)$$

$$(-\gamma_2 \cdot p_2 - m_2) \cosh V \psi = (\gamma_1 \cdot p_1 + m_1) \sinh V \psi , \quad (2.21b)$$

where, according to Eq. (2.5), $\psi(x^T)$ is the internal part of the wave function Ψ . Also, because of the constraint (2.4) (or (2.11)), the longitudinal components of the momentum operators are replaced by their eigenvalues determined in terms of P_L [Eq. (2.7)] and the masses :

$$p_{1L} = \frac{P_L}{2} + \frac{(m_1^2 - m_2^2)}{2P_L} , \quad (2.22a)$$

$$p_{2L} = \frac{P_L}{2} - \frac{(m_1^2 - m_2^2)}{2P_L} . \quad (2.22b)$$

By using the expression of V [Eq. (2.20)] and the properties of the γ -matrices (see Appendix B), one can bring the functions $\cosh V$ and $\sinh V$ to the left of the Dirac operators in Eqs. (2.21). If one defines $\bar{V} \equiv 2V_1 - V$, then, upon multiplying Eq. (2.21a) by $\cosh \bar{V}$ and Eq. (2.21b) by $-\sinh \bar{V}$ and subtracting the two equations from each other, one obtains a Dirac type equation for particle 1 as if it was placed in the presence of external potentials created by particle 2 :

$$\begin{aligned}
& \left\{ \left[\frac{P_L}{2} e^{2V_2} + \frac{(m_1^2 - m_2^2)}{2P_L} e^{-2V_2} \right] \gamma_{1L} - \frac{M}{2} e^{2V_1} - \frac{(m_1^2 - m_2^2)}{2M} e^{-2V_1} \right. \\
& + e^{-2U_4} \left[\gamma_1^T \cdot p^T + \frac{i\hbar}{2x^{T2}} (e^{-2T_4} - 1) (2\gamma_1^T \cdot x^T + i\gamma_1^{T\alpha} \sigma_{2\alpha\beta}^{TT} x^{T\beta}) + (e^{-2T_4} - 1) \frac{\gamma_1^T \cdot x^T}{x^{T2}} x^T \cdot p^T \right. \\
& \left. \left. - 2i\hbar e^{-2T_4} \gamma_2^T \cdot x^T (\dot{V}_1 + \gamma_{1L} \gamma_{2L} \dot{V}_2 + \gamma_{15} \gamma_{25} \dot{V}_3 + \gamma_1^T \cdot \gamma_2^T \dot{U}_4 + \frac{\gamma_1^T \cdot x^T \gamma_2^T \cdot x^T}{x^{T2}} \dot{T}_4) \right] \right\} \psi = 0 ,
\end{aligned} \tag{2.23}$$

where we have defined :

$$\dot{F} \equiv \frac{\partial F}{\partial x^{T2}} . \tag{2.24}$$

The Dirac type equation for particle 2 can be obtained from Eq. (2.23) by the replacements : $p_1 \leftrightarrow -p_2$, $x \rightarrow x$, $m_1 \leftrightarrow m_2$, $\gamma_1 \leftrightarrow \gamma_2$.

Equation (2.23) shows that the scalar potential acts essentially on the mass terms, the timelike vector potential acts on the energy terms, the spacelike vector potentials act on the transverse momentum dependent terms, while the pseudoscalar potential acts only through spin dependent terms. These results are consistent with the interpretation of Eq. (2.23) from the external potential standpoint. In particular, we did not find, in passing from Eqs. (2.21) to Eq. (2.23) new types of interaction. This is not a trivial property. Had we used a different parametrization than that provided by Eq. (2.17), for instance by decomposing \tilde{V} , rather than V , according to Eq. (2.20), we would have found that the original vector potentials induced in the analogue of Eq. (2.23) additional axial vector terms.

III. REDUCTION OF THE WAVE EQUATIONS TO THE PAULI-SCHRÖDINGER FORM

In order to bring Eqs. (2.21) into a more transparent form, we decompose the 4×4 matrix wave function on the basis of the matrices 1 , γ_L , γ_5 and $\gamma_L\gamma_5$ by defining 2×2 matrix components :

$$\psi = \psi_1 + \gamma_L\psi_2 + \gamma_5\psi_3 + \gamma_L\gamma_5\psi_4 \equiv \sum_{i=1}^4 \Gamma_i\psi_i . \quad (3.1)$$

Equations (2.21) are then projected in the above subspaces. We shall first consider the case of more general potentials than those introduced in Eqs. (2.20) and develop our method of calculation for this general case. They have the property of being functions of products of γ_1 and γ_2 matrices in equal number. This feature is obvious for one particle exchange diagrams (for parity conserving interactions), where the matrices γ_1 and γ_2 couple symmetrically to the propagator of the exchanged particle. However, when general vertex corrections are considered, one finds expressions in which some terms may not have the same number of γ_1 and γ_2 matrices, although globally the total expressions are symmetric in the exchanges $1 \leftrightarrow 2$. Such terms will not be considered here. Then, the most general (parity and time reversal invariant) potential we may consider has the following decomposition on the basis (3.1) :

$$V = v_1 + \gamma_{1L}\gamma_{2L} v_2 + \gamma_{15}\gamma_{25} v_3 + \gamma_{1L}\gamma_{15}\gamma_{2L}\gamma_{25} v_4 , \quad (3.2)$$

where the potentials v_i ($i = 1, \dots, 4$) may still have spin dependences.

Spin operators, which act in the 2×2 component subspaces, are defined by means of the Pauli-Lubanski operators :

$$\begin{aligned} W_{1S\alpha} &= -\frac{\hbar}{4}\epsilon_{\alpha\beta\mu\nu}P^\beta\sigma_1^{\mu\nu} , \quad W_{2S\alpha} = -\frac{\hbar}{4}\epsilon_{\alpha\beta\mu\nu}P^\beta\sigma_2^{\mu\nu} \quad (\epsilon_{0123} = +1) , \\ W_{1S}^2 &= W_{2S}^2 = -\frac{3}{4}\hbar^2 P^2 , \quad W_S = W_{1S} + W_{2S} . \end{aligned} \quad (3.3)$$

They also satisfy the relations :

$$\gamma_{1L}W_{1S\alpha} = \frac{\hbar P_L}{2}\gamma_{1\alpha}^T\gamma_{15} , \quad \gamma_{2L}W_{2S\alpha} = \frac{\hbar P_L}{2}\gamma_{2\alpha}^T\gamma_{25} . \quad (3.4)$$

(Note that the W 's are transverse vectors.)

We introduce the operators :

$$w = \left(\frac{2}{\hbar P_L}\right)^2 W_{1S} \cdot W_{2S} \xrightarrow{c.m.} -\frac{4}{\hbar^2} \mathbf{s}_1 \cdot \mathbf{s}_2 , \quad (3.5a)$$

$$w_{12} = \left(\frac{2}{\hbar P_L}\right)^2 \frac{W_{1S} \cdot x^T W_{2S} \cdot x^T}{x^{T2}} \xrightarrow{c.m.} -\frac{4}{\hbar^2} \frac{(\mathbf{s}_1 \cdot \mathbf{x})(\mathbf{s}_2 \cdot \mathbf{x})}{\mathbf{x}^2} \equiv -S_{12} . \quad (3.5b)$$

In the preceding relations we also displayed the corresponding c.m. expressions in terms of the usual spin operators.

Then, the potentials v_i of Eq. (3.2) can be decomposed as :

$$v_i = V_i + wU_i + w_{12}T_i \quad (i = 1, \dots, 4) , \quad (3.6)$$

where V_i , U_i and T_i are functions of x^{T2} and P_L .

For the combination of potentials we introduced in Eq. (2.20), we have to reexpress the transverse matrices γ^T according to Eqs. (3.4). By comparing Eq. (2.20) with Eqs. (3.2) and (3.6) we identify the indices used in Eq. (2.20).

In order to project Eqs. (2.21) onto the subspaces of the components ψ_i [Eq. (3.1)], it is useful to introduce projection matrices for these subspaces. These are :

$$\begin{aligned} \mathcal{P}_1 &= \frac{1}{4}(1 + \gamma_{1L}\gamma_{2L})(1 + \gamma_{15}\gamma_{25}) , & \mathcal{P}_2 &= \frac{1}{4}(1 + \gamma_{1L}\gamma_{2L})(1 - \gamma_{15}\gamma_{25}) , \\ \mathcal{P}_3 &= \frac{1}{4}(1 - \gamma_{1L}\gamma_{2L})(1 + \gamma_{15}\gamma_{25}) , & \mathcal{P}_4 &= \frac{1}{4}(1 - \gamma_{1L}\gamma_{2L})(1 - \gamma_{15}\gamma_{25}) . \end{aligned} \quad (3.7)$$

They satisfy the relations :

$$\mathcal{P}_i \mathcal{P}_j = \delta_{ij} \mathcal{P}_j , \quad \mathcal{P}_i \Gamma_j = \delta_{ij} \Gamma_j \quad (i, j = 1, \dots, 4) . \quad (3.8)$$

(The Γ 's are defined in Eq. (3.1).)

The general potential V [Eq. (3.2)] can be expressed in terms of the projection matrices \mathcal{P} :

$$\begin{aligned}
V &= \sum_{i=1}^4 a_i \mathcal{P}_i , \\
a_i &= A_i + w B_i + w_{12} C_i ;
\end{aligned} \tag{3.9}$$

the relations between the sets A_i , B_i , C_i and V_i , U_i , T_i [Eq. (3.6)] are easily established. For the potential (2.20) we shall consider below, we have the identifications :

$$\begin{aligned}
a_1 &= V_1 + V_2 + V_3 + w U_4 + w_{12} T_4 , \\
a_2 &= V_1 + V_2 - V_3 - w U_4 - w_{12} T_4 , \\
a_3 &= V_1 - V_2 + V_3 - w U_4 - w_{12} T_4 , \\
a_4 &= V_1 - V_2 - V_3 + w U_4 + w_{12} T_4 .
\end{aligned} \tag{3.10}$$

The projectors (3.7) are mainly needed for the evaluation of the exponential functions of V , present in Eqs. (2.21). One has the simple property :

$$\exp \left(\sum_{i=1}^4 a_i \mathcal{P}_i \right) = \sum_{i=1}^4 \mathcal{P}_i e^{a_i} . \tag{3.11}$$

Equations (2.21) then yield the eight coupled (compatible) equations for the four components ψ_i ($i = 1, \dots, 4$) :

$$\begin{aligned}
P_L e^{-a_3} \psi_3 - M e^{a_4} \psi_4 + \frac{2}{\hbar P_L} (W_{1S} - W_{2S}) \cdot p e^{-a_1} \psi_1 &= 0 , \\
P_L e^{-a_4} \psi_4 - M e^{a_3} \psi_3 - \frac{2}{\hbar P_L} W_S \cdot p e^{-a_2} \psi_2 &= 0 , \\
\frac{(m_1^2 - m_2^2)}{P_L} e^{-a_1} \psi_1 - M e^{a_2} \psi_2 + \frac{2}{\hbar P_L} W_S \cdot p e^{-a_3} \psi_3 &= 0 , \\
\frac{(m_1^2 - m_2^2)}{P_L} e^{-a_2} \psi_2 - M e^{a_1} \psi_1 - \frac{2}{\hbar P_L} (W_{1S} - W_{2S}) \cdot p e^{-a_4} \psi_4 &= 0 , \\
\frac{(m_1^2 - m_2^2)}{P_L} e^{a_3} \psi_3 - \frac{(m_1^2 - m_2^2)}{M} e^{-a_4} \psi_4 + \frac{2}{\hbar P_L} W_S \cdot p e^{a_1} \psi_1 &= 0 , \\
\frac{(m_1^2 - m_2^2)}{P_L} e^{a_4} \psi_4 - \frac{(m_1^2 - m_2^2)}{M} e^{-a_3} \psi_3 - \frac{2}{\hbar P_L} (W_{1S} - W_{2S}) \cdot p e^{a_2} \psi_2 &= 0 , \\
P_L e^{a_1} \psi_1 - \frac{(m_1^2 - m_2^2)}{M} e^{-a_2} \psi_2 + \frac{2}{\hbar P_L} (W_{1S} - W_{2S}) \cdot p e^{a_3} \psi_3 &= 0 , \\
P_L e^{a_2} \psi_2 - \frac{(m_1^2 - m_2^2)}{M} e^{-a_1} \psi_1 - \frac{2}{\hbar P_L} W_S \cdot p e^{a_4} \psi_4 &= 0 .
\end{aligned} \tag{3.12}$$

By using the property $(W_S.p)(W_{1S} - W_{2S}).p = 0$, one easily deduces from Eqs. (3.12) the following four equations :

$$\begin{aligned}
W_S.p (P_L e^{-a_3} \psi_3) &= W_S.p (M e^{a_4} \psi_4) , \\
(W_{1S} - W_{2S}).p (P_L e^{-a_4} \psi_4) &= (W_{1S} - W_{2S}).p (M e^{a_3} \psi_3) , \\
(W_{1S} - W_{2S}).p (P_L e^{a_2} \psi_2) &= (W_{1S} - W_{2S}).p \left(\frac{(m_1^2 - m_2^2)}{M} e^{-a_1} \psi_1 \right) , \\
W_S.p (P_L e^{a_1} \psi_1) &= W_S.p \left(\frac{(m_1^2 - m_2^2)}{M} e^{-a_2} \psi_2 \right) ,
\end{aligned} \tag{3.13}$$

which are useful during intermediate stages of calculation and for consistency checks.

Among the four components ψ_i , it is the combination $\psi_3 + \psi_4$ which is the dominant one in the nonrelativistic limit. The quantum numbers of the state are therefore determined by ψ_3 and ψ_4 . For this reason one must eliminate from Eqs. (3.12) ψ_1 and ψ_2 in terms of ψ_3 and ψ_4 and finally keep ψ_3 or ψ_4 . We shall keep the component ψ_3 and express the eigenvalue equation in terms of it; the calculations with ψ_4 parallel those with ψ_3 and we shall indicate at the end the substitutions to use to obtain the eigenvalue equation with ψ_4 .

The third and seventh of Eqs. (3.12) allow us to express ψ_1 and ψ_2 in terms of ψ_3 :

$$\begin{aligned}
\psi_1 = e^{(a_1 - 2h)} \frac{2}{\hbar P^2} \{ &- e^{-2a_1} (W_{1S} - W_{2S}).p e^{a_3} \psi_3 \\
&+ \frac{(m_1^2 - m_2^2)}{M^2} e^{-2(a_1 + a_2)} W_S.p e^{-a_3} \psi_3 \} ,
\end{aligned} \tag{3.14a}$$

$$\begin{aligned}
\psi_2 = e^{(a_2 - 2h)} \frac{2}{\hbar M P_L} \{ &+ e^{-2a_2} W_S.p e^{-a_3} \psi_3 \\
&- \frac{(m_1^2 - m_2^2)}{P^2} e^{-2(a_1 + a_2)} (W_{1S} - W_{2S}).p e^{a_3} \psi_3 \} ,
\end{aligned} \tag{3.14b}$$

where we have defined :

$$e^{2h} = 1 - \frac{(m_1^2 - m_2^2)^2}{M^2 P^2} e^{-2(a_1 + a_2)} \tag{3.15a}$$

$$= 1 - \frac{(m_1^2 - m_2^2)^2}{M^2 P^2} e^{-4(V_1 + V_2)} , \tag{3.15b}$$

the second equation, (3.15b), being valid only for the case of potential (2.20).

One then obtains two independent equations for ψ_3 and ψ_4 :

$$\begin{aligned}
MP_L e^{-a_4} \psi_4 &= M^2 e^{a_3} \psi_3 \\
&+ \left(\frac{2}{\hbar P_L}\right)^2 W_{S.p} e^{-2h} \{ e^{-2a_2} W_{S.p} e^{-a_3} \psi_3 \\
&- \frac{(m_1^2 - m_2^2)}{P^2} e^{-2(a_1 + a_2)} (W_{1S} - W_{2S}).p e^{a_3} \psi_3 \} , \quad (3.16a)
\end{aligned}$$

$$\begin{aligned}
MP_L e^{a_4} \psi_4 &= P^2 e^{-a_3} \psi_3 \\
&+ \left(\frac{2}{\hbar P_L}\right)^2 (W_{1S} - W_{2S}).p e^{-2h} \{ -e^{-2a_1} (W_{1S} - W_{2S}).p e^{a_3} \psi_3 \\
&+ \frac{(m_1^2 - m_2^2)}{M^2} e^{-2(a_1 + a_2)} W_{S.p} e^{-a_3} \psi_3 \} . \quad (3.16b)
\end{aligned}$$

Finally, eliminating ψ_4 through Eqs. (3.16) one obtains the eigenvalue equation with ψ_3 :

$$\begin{aligned}
P^2 e^{-(a_3 + a_4)} \psi_3 &- M^2 e^{(a_3 + a_4)} \psi_3 \\
&- \left(\frac{2}{\hbar P_L}\right)^2 e^{a_4} W_{S.p} e^{-2h} \{ e^{-2a_2} W_{S.p} e^{-a_3} \\
&- \frac{(m_1^2 - m_2^2)}{P^2} e^{-2(a_1 + a_2)} (W_{1S} - W_{2S}).p e^{a_3} \} \psi_3 \\
&- \left(\frac{2}{\hbar P_L}\right)^2 e^{-a_4} (W_{1S} - W_{2S}).p e^{-2h} \{ e^{-2a_1} (W_{1S} - W_{2S}).p e^{a_3} \\
&- \frac{(m_2^2 - m_1^2)}{M^2} e^{-2(a_1 + a_2)} W_{S.p} e^{-a_3} \} \psi_3 = 0 . \quad (3.17)
\end{aligned}$$

Had we expressed ψ_1 and ψ_2 in terms of ψ_4 and eliminated ψ_3 in terms of ψ_4 , we would have obtained the eigenvalue equation with ψ_4 , which can be read off Eq. (3.17) with the following substitutions *in the terms containing the spin operators* : $a_3 \leftrightarrow -a_4$ and $(m_1^2 - m_2^2)/P^2 \leftrightarrow (m_1^2 - m_2^2)/M^2$.

Equation (3.17) is a second order differential equation and has the form of a generalized Pauli-Schrödinger equation. It is possible to bring it into a more standard form by moving the momentum operators to the utmost right and having recourse to wave function transformations. We shall undertake these operations for the case when potential V takes the form (2.20) (or (3.10)) (scalar, pseudoscalar and vector potentials), the calculations, as well as the final expressions, being relatively simple. For simplicity of notation and a more transparent reading of the formulas, we shall express the resulting equation in the c.m. frame. We use the notations :

$$\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2, \quad \mathbf{L} = \mathbf{x} \times \mathbf{p}, \quad \mathbf{J} = \mathbf{L} + \mathbf{S}, \quad S_{12} = \frac{4}{\hbar^2} \frac{(\mathbf{s}_1 \cdot \mathbf{x})(\mathbf{s}_2 \cdot \mathbf{x})}{\mathbf{x}^2}. \quad (3.18)$$

Upon bringing, in Eq. (3.17) and for the case of potential (2.20) (or (3.10)), the momentum operators to the utmost right, one finds that the quadratic terms in \mathbf{p} sum up to yield, apart from a multiplicative potential dependent factor, the operator \mathbf{p}^2 . However, one also meets terms proportional to $i\hbar \mathbf{x} \cdot \mathbf{p}$. To get rid of the latter terms, it is necessary to use the change of function :

$$\psi_3 = e^{(2U_4 + 2T_4 + h)} \left\{ \frac{1}{2}(1 + S_{12}) e^{2V_1} + \frac{1}{2}(1 - S_{12}) e^{2V_2} \right\} \phi_3. \quad (3.19)$$

After some lengthy calculations (see Appendix B for useful formulas that are utilized for the spin and orbital angular momentum operators) one ends up with the following eigenvalue equation :

$$\begin{aligned}
& \left\{ e^{4(U_4 + T_4)} \left[\frac{P^2}{4} e^{4V_2} - \frac{M^2}{4} e^{4V_1} - \frac{(m_1^2 - m_2^2)^2}{4M^2} e^{-4V_1} + \frac{(m_1^2 - m_2^2)^2}{4P^2} e^{-4V_2} \right] \right. \\
& - \mathbf{p}^2 - 4\hbar^2 \mathbf{x}^2 (V'_1 + V'_2 + V'_3 - U'_4 + T'_4 + h')^2 \\
& + 6\hbar^2 (V'_1 + V'_2 + V'_3 - U'_4 + T'_4 + h') + 4\hbar^2 \mathbf{x}^2 (V''_1 + V''_2 + V''_3 - U''_4 + T''_4 + h'') \\
& - \mathbf{L}^2 \frac{1}{\mathbf{x}^2} (e^{4T_4} - 1) \\
& - 4\mathbf{S}^2 \left[(V'_1 + V'_2 + V'_3 + U'_4 + T'_4 + h') (1 + 4\mathbf{x}^2 U'_4) \right. \\
& \quad \left. - (U'_4 + 2\mathbf{x}^2 U''_4) - \frac{1}{4\mathbf{x}^2} (e^{4T_4} - 1) \right] \\
& + 4 \frac{(\mathbf{S} \cdot \mathbf{x})^2}{\mathbf{x}^2} \left[(V'_1 + V'_2 + 2U'_4 + T'_4 + h') (1 + 4\mathbf{x}^2 (V'_3 + U'_4)) \right. \\
& \quad \left. - 2\mathbf{x}^2 (V''_3 + U''_4) - \frac{1}{2\mathbf{x}^2} (e^{4T_4} - 1) \right] \\
& + 4 \left(\mathbf{L} \cdot \mathbf{S} + \mathbf{S}^2 - \frac{(\mathbf{S} \cdot \mathbf{x})^2}{\mathbf{x}^2} \right) e^{2T_4} \left[\frac{1}{\mathbf{x}^2} \sinh^2(V_1 - V_2) - \frac{1}{2\mathbf{x}^2} (e^{2T_4} - 1) \cosh 2(V_1 - V_2) \right. \\
& \quad \left. + (V'_1 + V'_2 + 2U'_4 + h') \cosh 2(V_1 - V_2) - (V'_1 - V'_2) \sinh 2(V_1 - V_2) \right] \\
& + 4 \left[\frac{(\mathbf{S} \cdot \mathbf{x})^2}{\hbar^2 \mathbf{x}^2}, \mathbf{L} \cdot \mathbf{S} \right] e^{2T_4} \left[\frac{1}{2\mathbf{x}^2} (e^{2T_4} - 2) \sinh 2(V_1 - V_2) \right. \\
& \quad \left. - (V'_1 + V'_2 + 2U'_4 + h') \sinh 2(V_1 - V_2) + (V'_1 - V'_2) \cosh 2(V_1 - V_2) \right] \\
& - 4(m_1^2 - m_2^2) \left(\frac{1}{P^2} + \frac{1}{M^2} \right) \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2) e^{2[T_4 - (V_1 + V_2) - h]} (V'_1 + V'_2) \\
& - 4(m_1^2 - m_2^2) \left(\frac{1}{P^2} - \frac{1}{M^2} \right) \left[\frac{(\mathbf{S} \cdot \mathbf{x})^2}{\hbar^2 \mathbf{x}^2}, \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2) \right] e^{2[T_4 - (V_1 + V_2) - h]} (V'_1 + V'_2) \Big\} \phi_3 \\
& = 0.
\end{aligned} \tag{3.20}$$

We recall that the functions V_1 , V_2 , V_3 , U_4 and T_4 correspond to the scalar (V_1), timelike vector (V_2), pseudoscalar (V_3) and spacelike vector (U_4 and T_4) potentials. The function h is defined in Eq. (3.15). We have also defined :

$$F' \equiv \frac{\partial F}{\partial \mathbf{x}^2}, \quad F'' \equiv \frac{\partial^2 F}{(\partial \mathbf{x}^2)^2}. \tag{3.21}$$

Equation (3.20) is the main result of the present paper. It corresponds to the reduction of the initial covariant wave equations (2.1), by means of the parametrizations (2.17) and (2.20) and of the wave function transformations and decomposition (2.18), (3.1), (3.19), to the Pauli-Schrödinger form, in which ϕ_3 is a 2×2 matrix wave function.

The above reduction can be realized with respect to the component ψ_4 as well. In this case the wave function transformation (3.19) is replaced by the following one :

$$\psi_4 = e^{(2U_4 + 2T_4 + h)} \left\{ \frac{1}{2}(1 + S_{12}) e^{2V_2} + \frac{1}{2}(1 - S_{12}) e^{2V_1} \right\} \phi_4 . \quad (3.22)$$

The equation satisfied by ϕ_4 is the same as Eq. (3.20), except a global change of sign in front of the terms proportional to the commutators $[(\mathbf{S} \cdot \mathbf{x})^2 / (\hbar^2 \mathbf{x}^2) , \mathbf{L} \cdot \mathbf{S}]$ and $[(\mathbf{S} \cdot \mathbf{x})^2 / (\hbar^2 \mathbf{x}^2) , \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2)]$.

In order to reconstitute the whole wave function ψ from ψ_3 , it is also necessary to calculate ψ_4 from ψ_3 (or conversely, ψ_3 from ψ_4 if the latter is first obtained). To this end, the first two of Eqs. (3.13) kinematically yield the desired relations in the sectors of states corresponding to the quantum numbers $j = \ell$. However, in the sectors with quantum numbers $j \neq \ell$ these equations do not allow one to obtain kinematically the relationship between ψ_3 and ψ_4 . It is therefore necessary to go back to the two coupled equations (3.16). Either of these equations can be used to express ψ_4 in terms of ψ_3 . As they stand, these equations involve second order differential operators. In order to obtain a relationship involving at most first order differential operators, one eliminates the operator \mathbf{p}^2 (see for this Eq. (B8)) by the use of Eq. (3.17). The relation giving ψ_4 in terms of ψ_3 is presented in Appendix B [Eq. (B9)] where we also explain how to express ψ_3 in terms of ψ_4 .

For completeness, we rewrite, in Appendix B [Eqs. (B10)], Eqs. (3.14) giving ψ_1 and ψ_2 in terms of ψ_3 , in more explicit forms when potential V has the form (2.20).

Equation (3.20) can be solved by decomposing the wave function ϕ_3 along radial wave functions and the generalized spherical harmonics, $\mathcal{Y}_{j\ell s}^m$, eigenfunctions of the operators $\mathbf{J}^2, \mathbf{J}_z, \mathbf{L}^2, \mathbf{S}^2$.

When all potentials are present, and in the unequal mass case, there are only two sectors of quantum numbers where ϕ_3 is an eigenfunction of the latter operators : i) $s = 0, \ell = 0, j = 0$; ii) $s = 1, \ell = 1, j = 0$. For the other sectors ϕ_3 is a combination of different eigenfunctions : $s = 0$ and $s = 1$ for the sectors with $\ell = j$; $\ell = |j \pm 1|$ for the sectors with $s = 1$. In the equal mass case, a simplification occurs, in that for the sectors with $\ell = j$,

the subsectors with $s = 0$ and $s = 1$ disentangle. A similar simplification also occurs in the unequal mass case when V_1 and V_2 are absent, or when $V_1 + V_2 = 0$ (or constant). Complete diagonalization occurs only for harmonic oscillators.

The presence in Eq. (3.20) of the commutators $[(\mathbf{S} \cdot \mathbf{x})^2 / (\hbar^2 \mathbf{x}^2) , \mathbf{L} \cdot \mathbf{S}]$ and $[(\mathbf{S} \cdot \mathbf{x})^2 / (\hbar^2 \mathbf{x}^2) , \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2)]$, might suggest that some of the energy eigenvalues are not real. These operators connect only off-diagonal matrix elements of the sectors $\ell = |j \pm 1|$. As is known, such matrix elements contribute only with the modulus squared of their values, and therefore the resulting energy eigenvalues are real. This also explains why, on solving either of Eq. (3.20) or the equivalent equation of ϕ_4 , where the signs in front of the above commutators are changed, one obtains the same energy eigenvalues.

In this connection, it is worth emphasizing that if, instead of considering the complete norm (2.19), where in any event the hermiticity property of the energy operator is manifest provided V is superficially hermitian, one tries to construct the scalar product or the norm for ϕ_3 directly from Eq. (3.20), one has to consider as the adjoint of ϕ_3 the function ϕ_4^\dagger rather than ϕ_3^\dagger , precisely because of the antihermitian nature of the abovementioned commutators. If Eq. (3.20) is written in compact form as $L_3 \phi_3 = 0$, then following standard methods [3,12,15], one ends up with a norm proportional to $\int d^3\mathbf{x} \text{Tr} \phi_4^\dagger \frac{\partial L_3}{\partial \mathbf{P}^2} \phi_3$, the hermitian conjugate of which is nothing but $\int d^3\mathbf{x} \text{Tr} \phi_3^\dagger \frac{\partial L_4}{\partial \mathbf{P}^2} \phi_4$, because $L_3^\dagger = L_4$, where L_4 is the wave equation operator of ϕ_4 , satisfying $L_4 \phi_4 = 0$.

Finally, we observe that when the potentials V_1 and V_2 are absent, then ϕ_3 and ϕ_4 satisfy the same eigenvalue equation, and yet they are not identical in general (see Eq. (B9)). This means that the energy spectrum has then degeneracies coming from the sectors with $\ell = |j \pm 1|$. These degeneracies increase in the case of potentials yielding harmonic oscillators.

IV. INFINITE MASS LIMIT

In this section we shall study the limit of Eq. (3.20) when one of the particles, 2, say, becomes infinitely massive. It is known that the Bethe-Salpeter equation does not have a simple behavior in this limit. The origin of this difficulty lies in the fact that the kernel of the Bethe-Salpeter equation contains only irreducible (crossed ladder) diagrams, which do not have suitable limits. In order to reproduce in this limit, from the Bethe-Salpeter equation, the one-particle Dirac equation in the presence of the potential created by the infinitely massive particle, one must take into account contributions coming from the infinite number of crossed ladder diagrams [17,18].

The situation is much simpler for the wave equations of Constraint Theory. Here, the potential is calculated from the diagrams of the off-mass shell scattering amplitude, to which one adds “constraint diagrams”, obtained from reducible diagrams with the use of constraint (2.4) [Eq. (2.10)]. To a given formal order of perturbation theory, the sum of all ladder and crossed ladder type Feynman diagrams (without vertex corrections) does have a simple limit : the sum of the n^{th} order diagrams (in coupling constant squared) contains the factor $\delta(k_{10}) \cdots \delta(k_{n-10})$, where the k ’s are the internal momenta of the loops ($n \geq 2$) [17,18]. This term is then cancelled by the dominant term of the “constraint diagrams” of n^{th} order [7] (see Appendix C). Therefore, in the limit $m_2 \rightarrow \infty$ it is only the one-particle exchange diagram (including vacuum polarization) which survives and leads to the one-particle Dirac equation with the corresponding static potential.

The expressions of the potentials calculated from such diagrams are presented in Appendix A. They essentially have the energy dependence of the form

$$V = \frac{1}{2P_L} V^{(0)} , \quad (4.1)$$

and, therefore, when m_2 goes to infinity their behavior is :

$$\lim_{m_2 \rightarrow \infty} V = \frac{1}{2m_2} V^{(0)} + O\left(\frac{1}{m_2}\right) . \quad (4.2)$$

The contributions coming from higher order diagrams are of order $O(1/m_2^2)$. If one replaces in Eq. (2.23) the various potentials by their leading behaviors (4.2), one immediately finds that Eq. (2.23) reduces to the Dirac equation of particle 1 in the presence of the external static scalar and timelike vector potentials $V_1^{(0)}$ and $V_2^{(0)}$. It is, however, also instructive to study the above limit in Eq. (3.20), where the contributions of each potential are more involved.

To this end we expand each potential V_a ($a = 1, \dots, 4$, V_4 representing here either U_4 or T_4) up to its next to the leading term:

$$V_a \simeq \frac{1}{2m_2} V_a^{(0)} + \frac{1}{m_2^2} W_a^{(1)}. \quad (4.3)$$

The total energy P_L is decomposed as:

$$P_L = m_2 + E. \quad (4.4)$$

The terms inside the brackets containing V_1 and V_2 have the following limit:

$$\lim_{m_2 \rightarrow \infty} \left[\frac{P^2}{4} e^{4V_2} - \frac{M^2}{4} e^{4V_1} - \frac{(m_1^2 - m_2^2)^2}{4M^2} e^{-4V_1} + \frac{(m_1^2 - m_2^2)^2}{4P^2} e^{-4V_2} \right] = \left[(E + V_2^{(0)})^2 - (m_1 + V_1^{(0)})^2 \right]. \quad (4.5)$$

Notice that the terms $W_a^{(1)}$ have disappeared, although the brackets were containing terms of the order of m_2^2 . It is also evident that the factor $\exp(4(U_4 + T_4))$ in front of the brackets in Eq. (3.20) will contribute as unity.

In the remaining terms of Eq. (3.20), besides the kinetic energy term, it is only those containing h' , h'' and $\exp(-2h)$ which will survive. They have the limits:

$$\begin{aligned} \lim_{m_2 \rightarrow \infty} \frac{2}{m_2} e^{-2h} &= [(E + V_2^{(0)}) + (m_1 + V_1^{(0)})]^{-1}, \\ \lim_{m_2 \rightarrow \infty} h' &= \frac{(V_2^{(0)'} + V_1^{(0)'})}{2 [(E + V_2^{(0)}) + (m_1 + V_1^{(0)})]}. \end{aligned} \quad (4.6)$$

After averaging over the spin states of particle 2, one ends up with the following equation (where φ designates the limit of ϕ_3/m_2):

$$\left\{ (E + V_2^{(0)})^2 - \mathbf{p}^2 - (m_1 + V_1^{(0)})^2 - 4\hbar^2 \mathbf{x}^2 h'^2 + 6\hbar^2 h' + 4\hbar^2 \mathbf{x}^2 h'' + 8h' \mathbf{L} \cdot \mathbf{s}_1 \right\} \varphi = 0 . \quad (4.7)$$

This equation is nothing but the reduced form of the Dirac equation in the presence of the static scalar and timelike vector potentials after the decomposition $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ and the change of function $\psi_+ = (E + V_2^{(0)} + m_1 + V_1^{(0)})^{1/2} \varphi$ are used (in the basis where γ_0 is diagonal). The latter is the limit (up to a multiplicative constant factor proportional to $m_2^{-1/2}$), when m_2 goes to infinity, of the change of function (3.19).

V. NONRELATIVISTIC LIMIT TO ORDER $1/c^2$

We study, in this section, the nonrelativistic limit of Eq. (3.20) to order $1/c^2$, the gauge transformation property of the resulting Hamiltonian for the electromagnetic interaction case and the structure of confining scalar potentials.

A. General case in perturbation theory

To recover the explicit c dependence of the various terms in Eq. (3.20) (where c had been set equal to 1) we must adjoin a c factor to each of the terms \hbar , \mathbf{p} , \mathbf{s}_1 , \mathbf{s}_2 , \mathbf{L} and a c^2 factor to m_1 and m_2 . On the other hand, at leading order, the potentials have, in perturbation theory, the dependence on P_L of the type $1/(2P_L)$ (see Appendix A). Therefore, we expand the potentials as follows :

$$V_a = \frac{1}{2P_L} V_a^{(0)} + \frac{1}{2P_L^2} V_a^{(1)} + O(P_L^{-3}) \quad (a = 1, \dots, 4) . \quad (5.1)$$

Here, $V_a^{(0)}/(2P_L)$ is the contribution of the one-particle exchange diagram (or an effective representation of it) and $V_a^{(1)}/(2P_L^2)$ is the leading contribution coming from the higher order diagrams. (In the perturbation theory of the present formalism, it is the two-particle exchange diagrams which provide the latter term.)

To carry out the expansion, we split the total energy into the total mass and the binding energy E :

$$P_L = M + E . \quad (5.2)$$

The term E , which is an eigenvalue, is brought to the utmost right and replaced there by the Hamiltonian H . In expressions containing the term H^2 , which enters only in $O(c^{-2})$ quantities, one replaces H by its nonrelativistic expression.

The resulting Hamiltonian has the the following expression (in the c.m. frame) :

$$\begin{aligned}
H = & \frac{\mathbf{p}^2}{2\mu} + V_1^{(0)} - V_2^{(0)} - \frac{1}{8} \left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) (\mathbf{p}^2)^2 \\
& - \frac{1}{2m_1m_2} [\mathbf{p}^2, U_4^{(0)} + T_4^{(0)}]_+ - \frac{(m_1^2 + m_2^2)}{4m_1^2m_2^2} [\mathbf{p}^2, V_1^{(0)}]_+ \\
& - \frac{1}{2M} (V_1^{(0)} - V_2^{(0)})^2 + \frac{1}{M} (V_1^{(1)} - V_2^{(1)}) \\
& - \frac{\hbar^2}{4m_1m_2} \Delta \left[V_1^{(0)} + V_2^{(0)} + V_3^{(0)} - U_4^{(0)} + T_4^{(0)} + \frac{(m_1 - m_2)^2}{2m_1m_2} (V_1^{(0)} + V_2^{(0)}) \right] \\
& + \frac{1}{m_1m_2} \mathbf{L}^2 \frac{T_4^{(0)}}{\mathbf{x}^2} + \frac{1}{m_1m_2} \mathbf{S}^2 \left[-\frac{1}{2} \Delta U_4^{(0)} + V_3^{(0)'} + U_4^{(0)'} + T_4^{(0)'} \right] \\
& - \frac{1}{m_1m_2} \frac{(\mathbf{S} \cdot \mathbf{x})^2}{\mathbf{x}^2} \left[-\frac{1}{2} \Delta (V_3^{(0)} + U_4^{(0)}) + 3(V_3^{(0)'} + U_4^{(0)'} + T_4^{(0)'} - \frac{T_4^{(0)}}{\mathbf{x}^2}) \right] \\
& - \frac{1}{m_1m_2} \mathbf{L} \cdot \mathbf{S} \left[V_1^{(0)'} + V_2^{(0)'} + 2U_4^{(0)'} - \frac{T_4^{(0)}}{\mathbf{x}^2} + \frac{(m_1 - m_2)^2}{2m_1m_2} (V_1^{(0)'} + V_2^{(0)'}) \right] \\
& - \frac{1}{2} \left(\frac{1}{m_1^2} - \frac{1}{m_2^2} \right) \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2) (V_1^{(0)'} + V_2^{(0)'}) .
\end{aligned} \tag{5.3}$$

Here, $\mu = m_1m_2/M$, ΔV is the Laplacian of V , $\Delta V = 6V' + 4\mathbf{x}^2V''$, and the derivatives are with respect to \mathbf{x}^2 . $[,]_+$ is the anticommutator. To obtain Eq. (5.3) we also used the wave function transformation :

$$\phi_3 = \left[1 - \frac{1}{M} (U_4^{(0)} + T_4^{(0)}) - \frac{1}{2M} V_1^{(0)} - \frac{(m_1^2 + m_2^2 - m_1m_2)}{2m_1m_2M} V_2^{(0)} \right] \phi , \tag{5.4}$$

to bring the Hamiltonian into a manifestly hermitian form. The term containing $[(\mathbf{S} \cdot \mathbf{x})^2 / (\hbar^2 \mathbf{x}^2), \mathbf{L} \cdot \mathbf{S}]$ in Eq. (3.20), which is formally of order c^{-2} , does not actually contribute, for it vanishes in diagonal matrix elements of the eigenvalues of the nonrelativistic Hamiltonian.

B. Electromagnetic interaction

We now consider the case of the electromagnetic interaction. It is known that bound state energies are gauge invariant [19,20]. We check this property to order c^{-2} . We set in Eq. (5.3) $V_1 = V_3 = 0$. Furthermore, considering the photon propagator [Eqs. (A6) and (A7)] we deduce that U_4 must be split into two terms, V_2 and a pure gauge term \tilde{U} ; T_4 is then equal to $2\mathbf{x}^2\tilde{U}'$:

$$U_4 = V_2 + \tilde{U} , \quad T_4 = 2\mathbf{x}^2 \tilde{U}' . \quad (5.5)$$

After this splitting is introduced, the Hamiltonian (5.3) for the electromagnetic interaction takes the form :

$$\begin{aligned} H = & \frac{\mathbf{p}^2}{2\mu} - V_2^{(0)} - \frac{1}{8} \left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) (\mathbf{p}^2)^2 \\ & - \frac{1}{m_1 m_2} (V_2^{(0)} + \tilde{U}^{(0)}) \mathbf{p}^2 - \frac{2\tilde{U}^{(0)'}}{m_1 m_2} x_i x_j p_i p_j \\ & + \frac{2i\hbar}{m_1 m_2} (V_2^{(0)'} + 2\tilde{U}^{(0)'}) \mathbf{x} \cdot \mathbf{p} + \frac{i\hbar}{m_1 m_2} \Delta \tilde{U}^{(0)} \mathbf{x} \cdot \mathbf{p} \\ & + \frac{\hbar^2}{4m_1 m_2} \Delta \left[\frac{(6m_1 m_2 - m_1^2 - m_2^2)}{2m_1 m_2} V_2^{(0)} + 3\tilde{U}^{(0)} \right] \\ & + \frac{\hbar^2}{2m_1 m_2} \Delta (\mathbf{x}^2 \tilde{U}^{(0)'}) - \frac{1}{2M} (V_2^{(0)})^2 - \frac{1}{M} V_2^{(1)} \\ & + \frac{1}{m_1 m_2} \mathbf{S}^2 \left(-\frac{1}{2} \Delta V_2^{(0)} + V_2^{(0)'} \right) - \frac{1}{m_1 m_2} \frac{(\mathbf{S} \cdot \mathbf{x})^2}{\mathbf{x}^2} \left(-\frac{1}{2} \Delta V_2^{(0)} + 3V_2^{(0)'} \right) \\ & - \frac{1}{2} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} + \frac{4}{m_1 m_2} \right) \mathbf{L} \cdot \mathbf{S} V_2^{(0)'} - \frac{1}{2} \left(\frac{1}{m_1^2} - \frac{1}{m_2^2} \right) \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2) V_2^{(0)'} . \end{aligned} \quad (5.6)$$

We notice that the gauge function $\tilde{U}^{(0)}$ has disappeared from the spin dependent terms. To prove gauge invariance of the energy corresponding to the Hamiltonian (5.6), we must show that a unitary wave function transformation removes the arbitrary gauge function $\tilde{U}^{(0)}$ from H and brings it into its form of the Feynman gauge (in which $\tilde{U}^{(0)} = 0$), which is taken here as the reference gauge.

It can be checked that the following wave function transformation

$$\phi \simeq (1 + iG) \phi_F ,$$

$$G = \frac{1}{2\hbar M} (\tilde{U}^{(0)} \mathbf{p} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{p} \tilde{U}^{(0)}) \simeq -\frac{2\mu}{4i\hbar^2 M} [H , \int^{\mathbf{x}^2} \tilde{U}^{(0)} d\mathbf{x}^2] , \quad (5.7)$$

where ϕ_F is the wave function in the Feynman gauge, removes from Eq. (5.6) the existing $\tilde{U}^{(0)}$ dependent terms. [Actually, it can be shown, by using the connection of the Constraint Theory wave function with the Bethe-Salpeter wave function [7], that transformation (5.7) is the three-dimensional reduction, to the leading order of the present approximation, of the gauge transformation of the Bethe-Salpeter wave-function [21].]

The transformation (5.7) makes appear, however, in Eq. (5.6), a new $\tilde{U}^{(0)}$ dependent term, coming from the commutation of G with $V_2^{(0)}$ and equal to $2\mathbf{x}^2 \tilde{U}^{(0)} V_2^{(0)'} / M$. This term must be compensated by the gauge dependence of the second order term $V_2^{(1)}$, present in Eq. (5.6). If we designate by $V_{2F}^{(1)}$ the expression of $V_2^{(1)}$ in the Feynman gauge, then $V_2^{(1)}$ is related to V_{2F} by the relation:

$$V_2^{(1)} = V_{2F}^{(1)} + 2\mathbf{x}^2 \tilde{U}^{(0)} V_2^{(0)'} . \quad (5.8)$$

The expression of $V_{2F}^{(1)}$ was calculated from the two photon exchange diagrams by Rizov, Todorov and Aneva [22] in the Quasipotential Theory for spin- 1/2 - spin- 0 particle systems. The perturbation theory in the Quasipotential Theory is very similar to that of Constraint Theory [Eq. (2.10)], except that in the latter case off-mass shell scattering amplitudes are considered and nowhere extrapolations from on-shell quantities are used. These differences do not, however, show up at $O(\alpha^4)$. Furthermore, V_2 is a spin independent quantity to this order. One finds $V_{2F}^{(1)} = -(V_2^{(0)})^2$. (We also checked this result from Eq. (2.10) in the two-fermion case.) One then obtains for $V_2^{(1)}$ the following expression:

$$V_2^{(1)} = -(V_2^{(0)})^2 + 2\mathbf{x}^2 \tilde{U}^{(0)} V_2^{(0)'} . \quad (5.9)$$

Therefore, the Hamiltonian (5.6), together with the expression (5.9) of $V_2^{(1)}$, provides a gauge invariant wave equation to order c^{-2} . Its expression in the Feynman gauge is obtained by setting in Eq. (5.6) $\tilde{U}^{(0)} = 0$.

The Breit Hamiltonian is obtained by using the Coulomb gauge. In the present three-dimensional formalism and in the c.m frame, the expressions of the photon propagator in the Coulomb gauge and in the Landau gauge coincide in lowest order [Eq. (A10)]. The latter corresponds to the choice $\tilde{U}^{(0)} = -V_2^{(0)}/2$. By replacing $\tilde{U}^{(0)}$ with this expression in Eqs. (5.6) and (5.9), one recovers the Breit Hamiltonian [23]. (Notice that the term $\Delta\tilde{U}^{(0)}\mathbf{x}\cdot\mathbf{p}$ vanishes, for $\Delta\tilde{U}^{(0)}$ is now a δ -function. For the same reason the term $\Delta V_2^{(0)}(\mathbf{S}\cdot\mathbf{x})^2/\mathbf{x}^2$ can be replaced by $\Delta V_2^{(0)}\mathbf{S}^2/3$.) In this gauge all quadratic terms in $V_2^{(0)}$ (replaced by its Coulombic expression (A9)) disappear. This explains, once more, why in the Coulomb gauge one obtains the correct $O(\alpha^4)$ effects with one photon exchange diagrams only.

The reconstitution of a covariant electromagnetic potential from the previous results is, however, model dependent and sensitive to implicit assumptions made about the contributions of the multiphoton exchange diagrams. A first model is provided by the Todorov potential [14], introduced in Quasipotential Theory and later used in spectroscopic calculations (for electromagnetic as well as for short distance quark interactions) by Crater, Van Alstine and collaborators [2,4]. In our notations, it corresponds to the choice (in the Feynman gauge) :

$$\begin{aligned} V_2 &= U_4 = \frac{1}{4} \ln \left(1 + \frac{2V_2^{(0)}}{P_L} \right), & T_4 &= 0, \\ V_2^{(0)} &= \frac{\alpha}{r}, & r &= \sqrt{-x^2}. \end{aligned} \quad (5.10)$$

It was shown in Ref. [24] that for large values of the coupling constant ($\alpha \sim 1/2$) this potential leads to an instability of the vacuum state and might play a role in mechanisms of spontaneous chiral symmetry breaking.

A second potential was also considered in Ref. [24]; it corresponds to the choice (in the Feynman gauge) :

$$V_2 = U_4 = \frac{1}{2} \frac{V_2^{(0)}}{(P_L + V_2^{(0)})}, \quad T_4 = 0, \quad (5.11)$$

$V_2^{(0)}$ being the same as in Eq. (5.10). This potential has the property of being regular at the origin and does not lead to an instability of the vacuum state.

C. Scalar potential

We now turn to the case of scalar potentials and, for simplicity, shall ignore the other potentials. We first consider exchange of massless particles. $V_1^{(0)}$ has then a Coulombic expression (A4). The term $V_1^{(1)}$ can be calculated in perturbation theory from the two-particle exchange diagrams [Eq. (2.10)]. One finds :

$$V_1^{(1)} = (V_1^{(0)})^2 , \quad (5.12)$$

with $V_1^{(0)}$ given in Eq. (A4). (The corresponding calculations will be reported elsewhere.) As in the electromagnetic case, it is possible to modify the wave function representation by means of the inverse of the transformation (5.7), with $\tilde{U}^{(0)}$ now replaced by $V_1^{(0)}/2$, and to reach a Coulomb-gauge like representation. It can be checked that in the resulting Hamiltonian all the quadratic terms in the potential disappear. Also, the momentum dependent part of the effective potential then coincides with that obtained by Olsson and Miller [25], who, with a different method took into account the retardation effects. (For a discussion see also Ref. [26].)

The foregoing results lead us to the general conclusion that, when covariant propagators are used, it is necessary to evaluate the leading contribution coming from the two-particle exchange diagrams, in order to obtain the correct $O(c^{-2})$ effects. It is only in the Coulomb (or Landau) gauge for the electromagnetic interaction or in a particular representation of the wave function for the scalar interaction that the above evaluation may be circumvented.

As far as the spin dependent terms are concerned, Hamiltonian (5.3) agrees with the general expressions obtained previously in the literature [27].

Scalar potentials play also an important role in the description of interquark confining interactions. Confining potentials, however, do not correspond to simple diagrams of perturbation theory and therefore they may display complicated or nontrivial energy dependences, other than those of Eq. (5.1), implied by the underlying dynamics. These energy dependences do not affect, to order $1/c^2$, the spin dependent part of the Hamiltonian, but modify its momentum dependent part.

To analyze the structure of the Hamiltonian corresponding to a confining scalar potential, we shall adopt for the latter a more general energy dependence than that given in Eq. (5.1) by perturbation theory :

$$\begin{aligned} V_1 &= \frac{1}{2M} \tilde{V}_1^{(0)} \Big|_{P_L=M} + \frac{(P_L - M)}{2M} \frac{\partial \tilde{V}_1^{(0)}}{\partial P_L} \Big|_{P_L=M} + \frac{V_1^{(1)}}{2M^2} + O(c^{-6}) , \\ \tilde{V}_1^{(0)} \Big|_{P_L=M} &= V_1^{(0)} . \end{aligned} \quad (5.13)$$

Here, $V_1^{(0)}$ is the nonrelativistic confining potential, while the second term in the right-hand side of the first equation takes into account the leading energy dependence of the relativistic potential V_1 . As in the perturbative case, $V_1^{(1)}$ is assumed to be a quadratic function of $V_1^{(0)}$ and presumably arises from the expansion of the relativistic potential V_1 in terms of $V_1^{(0)}$.

The expression of the Hamiltonian is dependent on the wave function representation, which is modified by transformations of the type (5.7). The latter are equivalent to the introduction of pure gauge vector potentials. These are described by the potentials U_4 and T_4 , the leading terms of which are parametrized as :

$$U_4 = \frac{1}{2M} U^{(0)} , \quad T_4 = \frac{1}{2M} T^{(0)} ; \quad (5.14)$$

they satisfy the second of Eqs. (5.5) :

$$T^{(0)} = 2\mathbf{x}^2 U^{(0)'} . \quad (5.15)$$

Notice, however, that $U^{(0)}$ and $T^{(0)}$ may still depend on ratios of masses, not present in perturbation theory.

As shown in subsection B, pure gauge vector potentials do not contribute to the spin dependent terms (to order $1/c^2$).

We shall ignore in our analysis the short distance vector potential, which has the structure of the electromagnetic potential, discussed in subsection B, and does not affect the leading large distance properties of the confining potential.

The resulting Hamiltonian takes the form (in the c.m. frame) :

$$\begin{aligned}
H = & \frac{\mathbf{p}^2}{2\mu} + V_1^{(0)} - \frac{1}{8} \left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) (\mathbf{p}^2)^2 \\
& - \frac{1}{4} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} - \frac{1}{m_1 m_2} \right) [\mathbf{p}^2, V_1^{(0)}]_+ + \frac{M}{4m_1 m_2} [\mathbf{p}^2, \frac{\partial \tilde{V}_1^{(0)}}{\partial P_L}]_+ \\
& - \frac{1}{2m_1 m_2} [\mathbf{p}^2, U^{(0)} + T^{(0)}]_+ + \frac{1}{m_1 m_2} \frac{T^{(0)}}{\mathbf{x}^2} \mathbf{L}^2 \\
& - \frac{\hbar^2}{8} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \Delta V_1^{(0)} + \frac{\hbar^2}{4m_1 m_2} \Delta (U^{(0)} - T^{(0)}) \\
& + \frac{1}{M} V_1^{(1)} + \frac{1}{2M} (V_1^{(0)})^2 + V_1^{(0)} \frac{\partial \tilde{V}_1^{(0)}}{\partial P_L} - \left(\frac{\mathbf{L} \cdot \mathbf{s}_1}{m_1^2} + \frac{\mathbf{L} \cdot \mathbf{s}_2}{m_2^2} \right) V_1^{(0)'} .
\end{aligned} \tag{5.16}$$

(The prime represents derivation with respect to \mathbf{x}^2 and Δ is the Laplacian; $[\ , \]_+$ is the anticommutator.)

A possible comparison of this Hamiltonian with other expressions obtained from quantum field theoretic calculations would permit the determination of its various pieces and, with reasonable assumptions, might lead to a reconstitution of the covariant expression of the confining potential.

In this respect, we mention the work of Barchielli, Montaldi and Prosperi [28], who, continuing the Wilson loop approach to the confining potential [29], as was developed earlier by Eichten and Feinberg [30] and by Gromes [31], obtained from QCD an expression for the Hamiltonian to order $1/c^2$, including its momentum dependent part.

The complete reconstitution of the confining potential from the comparison of the Hamiltonian (5.16) with that of Ref. [28] goes beyond the scope of the present paper and is left for future work. We simply sketch here the qualitative features obtained from such a comparison. As was already pointed out by Olsson and collaborators [32], the Hamiltonian of Ref. [28] favors the interpretation of the underlying dynamics with the semiclassical flux tube picture. Actually, this result is not very surprising, since one of the initial hypotheses used in Ref. [28], namely the dominance of the longitudinal color electric field, is itself motivated, at least partly, from that picture.

The expression of the relativistic confining potential V_1 suggested by the above compari-

son, turns out to be momentum dependent, typical of flux tube models. This means that in x -space V_1 should be represented by an integral operator (in the three-dimensional variable x^T). However, in the classical case, the momentum dependent part is a smooth function, and therefore, as a first approximation, one can use for it a mean value approximation, thus representing V_1 by a local function in x^T .

We also notice that the momentum dependent part of the Hamiltonian of Ref. [28] does not vanish in the limit $m_2 \rightarrow \infty$. Therefore, in this limit, particle 1 will satisfy a Dirac equation with still a momentum dependent potential.

VI. SUMMARY AND CONCLUDING REMARKS

We showed that the two-fermion relativistic wave equations of Constraint Theory can be reduced, by expressing the components of the 4×4 matrix wave function in terms of one of the 2×2 components, to a single equation of the Pauli-Schrödinger type, valid for all sectors of quantum numbers. This equation, which is relativistic invariant, can be analyzed and solved with the usual tools of nonrelativistic quantum mechanics - a feature which considerably simplifies the treatment of two-fermion relativistic problems.

The interaction potentials that are present in this equation belong to the general classes of scalar, pseudoscalar and vector interactions and are calculable in perturbation theory from Feynman diagrams. They are of the quasipotential type, in the sense that even in local approximations, which we adopted throughout this work, they may still depend on the c.m. energy of the system. This property allows one to take into account leading contributions of nonlocal effects.

In the limit when one of the masses becomes infinite, the equation reduces to the two-component form of the one-particle Dirac equation with external static potentials.

The Hamiltonian of the system to order $1/c^2$ reproduces most the known theoretical results obtained by other methods. Furthermore, in the electromagnetic case, gauge invariance of the wave equation is checked to this order, by considering the photon propagator in arbitrary covariant gauges.

A last application was devoted to the analysis of the structure of the Hamiltonian, to order $1/c^2$, in the case of confining interactions. We emphasized here the role of the c.m. energy dependence of the relativistic potential. We displayed, for confining scalar potentials, the general expression of the corresponding Hamiltonian, in the presence of pure gauge vector potentials. This expression can be used for comparison with other theoretical evaluations and reconstitution of the relativistic confining potential.

The relativistic invariance of the reduced Pauli-Schrödinger type equation also allows one to consider ultra-relativistic limits, which play a crucial role in particle physics.

APPENDIX A: THE POTENTIALS FROM ONE-PARTICLE EXCHANGE DIAGRAMS

In this appendix we display the relationships of the various potentials, introduced in Sec. II and used throughout this paper, with the propagators of the exchanged particles in perturbation theory. According to formula (2.10), the three-dimensional reduction of a propagator with mass μ results, in momentum space, from the operation :

$$\widetilde{D}(q^T, \mu) = D(q_L = 0, q^T, \mu) , \quad (\text{A1})$$

or in x -space :

$$\widetilde{D}(x^T, \mu) = \int dx_L D(x, \mu) , \quad (\text{A2})$$

D representing here the usual four-dimensional propagator. For scalar particles and in lowest order of perturbation theory the propagator is $D(q, \mu) = i/(q^2 - \mu^2 + i\epsilon)$.

For scalar interactions one has :

$$V_1(x^T) = (-i)^2 \frac{i}{2P_L} g_1 g_2 \widetilde{D}(x^T, \mu) , \quad (\text{A3})$$

g_1 and g_2 being the coupling constants of the external particles to the exchanged particle with mass μ . In lowest order :

$$V_1(x^T) = -\frac{1}{2P_L} \frac{g_1 g_2}{4\pi} \frac{e^{-\mu r}}{r} , \quad r = \sqrt{-x^{T2}} . \quad (\text{A4})$$

For pseudoscalar interactions, the potential V_3 has the same type of definitions as V_1 .

For QED one has :

$$D_{\mu\nu}(k) = -(g_{\mu\nu} D(k, 0) - k_\mu k_\nu F(k)) , \quad (\text{A5})$$

where the gauge function F is taken as a covariant function of k . Then :

$$\begin{aligned} \widetilde{D}_{\mu\nu}(x^T) &= -(g_{\mu\nu} \widetilde{D}(x^T, 0) + \partial_\mu^T \partial_\nu^T \widetilde{F}(x^T)) \\ &= -(g_{\mu\nu} \widetilde{D} + 2g_{\mu\nu}^{TT} \ddot{\widetilde{F}} + 4x_\mu^T x_\nu^T \ddot{\widetilde{F}}) , \end{aligned} \quad (\text{A6})$$

the dot operation being defined in Eq. (2.24).

By identification with Eq. (2.20) we find :

$$\begin{aligned} U_4 &= V_2 + \tilde{U} , \\ T_4 &= 2x^{T2} \dot{\tilde{U}} . \end{aligned} \tag{A7}$$

For particles with opposite charges we have :

$$V_2 = -(-i)^2 \frac{i}{2P_L} e^2 \tilde{D}(x^T, 0) . \tag{A8}$$

In lowest order, V_2 becomes :

$$V_2 = \frac{1}{2P_L} \frac{\alpha}{r} , \quad r = \sqrt{-x^{T2}} , \quad \alpha = \frac{e^2}{4\pi} . \tag{A9}$$

(Notice that the sign of the charge of the antiparticle is taken into account by the matrices γ'_2 of Eq. (2.20) which act on the wave function from the right. In the nonrelativistic limit, the matrix γ_{2L} has eigenvalue -1 .)

For covariant gauges described by a parameter ξ , we have in lowest order :

$$\tilde{D}_{\mu\nu}(x^T) = -\left(g_{\mu\nu} - g_{\mu\nu}^{TT} \frac{\xi}{2} + \frac{x_\mu^T x_\nu^T}{x^{T2}} \frac{\xi}{2} \right) \tilde{D}(x^T, 0) . \tag{A10}$$

The Feynman gauge is obtained with $\xi = 0$ and the Landau gauge with $\xi = 1$.

For vector interactions resulting from the exchange of a vector particle with mass μ , the function \tilde{F} in Eq. (A6) should be replaced by \tilde{D}/μ^2 .

As a final remark, we emphasize that relationships like (A3) and (A8) between the potentials and the exchanged particle propagators are *not* equivalent to the instantaneous approximation. The perturbation theory expansion of Eq. (2.10) always reproduces for the potential corresponding to a one-particle exchanged diagram (including vacuum polarization effects) a local expression in x^T , together with the energy factor $1/(2P_L)$. In the present formalism, the instantaneous approximation produces a nonlocal potential due to the presence of the momentum operator p^{T2} in it [7]. On the other hand, the instantaneous approximation does not provide the correct infinite mass limit, neither the correct $1/c^2$ expansion.

APPENDIX B: USEFUL FORMULAS FOR THE γ -MATRICES AND THE SPIN AND ORBITAL ANGULAR MOMENTUM OPERATORS

In this appendix we present formulas for the γ -matrices and the spin and orbital angular momentum operators that are used during the reduction operations from Eqs. (2.21) to Eq. (3.20). We also exhibit the relationships between the components ψ_1 , ψ_2 and ψ_4 with ψ_3 [Eq. (3.1)].

The transition from Eqs. (2.21) to Eqs. (2.23) is obtained with the use of the following properties of the γ -matrices :

$$e^{-\lambda\overline{V}} (\gamma_{1\alpha}^T \pm \gamma_{2\alpha}^T) e^{\lambda V} = e^{\pm 2\lambda U_4} [(\gamma_{1\alpha}^T \pm \gamma_{2\alpha}^T) + (\gamma_1^T \cdot x^T \pm \gamma_2^T \cdot x^T) \frac{1}{x^{T2}} (e^{\pm 2\lambda T_4} - 1) x_\alpha^T] , \quad (\text{B1})$$

where V is defined in Eq. (2.20) and \overline{V} is defined as :

$$\overline{V} = 2V_1 - V . \quad (\text{B2})$$

Equations (B1) are easily established by differentiating with respect to the parameter λ .

We notice that all the bilinear forms of the γ -matrices present in V [Eq. (2.20)] commute with each other.

Similarly one establishes the following relation giving the transformation law of p^T under the action of the exponential of V :

$$\begin{aligned} e^{-\lambda V} p_\alpha^T e^{\lambda V} &= p_\alpha^T + 2i\hbar\lambda [\dot{V}_1 + \gamma_{15}\gamma_{25}\dot{V}_3 \\ &+ \gamma_1^\mu\gamma_2^\nu (g_{\mu\nu}^{LL}\dot{V}_2 + g_{\mu\nu}^{TT}\dot{U}_4 + \frac{x_\mu^T x_\nu^T}{x^{T2}}\dot{T}_4)] x_\alpha^T \\ &+ \frac{i\hbar}{2x^{T2}} [(\sinh 2\lambda T_4) (\gamma_{1\alpha}^T \gamma_2^T \cdot x^T + \gamma_{2\alpha}^T \gamma_1^T \cdot x^T) \\ &+ i(\cosh 2\lambda T_4 - 1) (\sigma_{1\alpha\beta}^{TT} + \sigma_{2\alpha\beta}^{TT}) x^{T\beta} \\ &- 2(\sinh 2\lambda T_4) \frac{1}{x^{T2}} \gamma_1 \cdot x^T \gamma_2 \cdot x^T x_\alpha^T] , \end{aligned} \quad (\text{B3})$$

where the dot operation is defined in Eq. (2.24).

The transition from Eq. (3.17) to Eq. (3.20) uses the following properties of the spin operators, defined in Eqs. (3.3)-(3.5) :

$$\begin{aligned}
e^{-A+wB} (W_{1S} \pm W_{2S}).p e^{A+wB} &= e^{\mp 2B} [(W_{1S} \pm W_{2S}).p \\
&\quad + 2i\hbar (\dot{A} \mp 2\dot{B} - w\dot{B}) (W_{1S} \pm W_{2S}).x^T] , \\
e^{-A+wB} (W_{1S} \pm W_{2S})_\alpha e^{A+wB} &= e^{\mp 2B} (W_{1S} \pm W_{2S})_\alpha , \\
e^{w_{12}C} (W_{1S} \pm W_{2S})_\alpha e^{w_{12}C} &= (W_{1S} \pm W_{2S})_\alpha \\
&\quad - \frac{1}{x^{T2}} (1 - e^{\mp 2C}) (W_{1S} \pm W_{2S}).x x_\alpha^T , \\
e^{-w_{12}C} p_\alpha^T e^{w_{12}C} &= p_\alpha^T + \frac{i\hbar}{4x^{T2}} \left(\frac{2}{\hbar P_L} \right)^2 \left\{ (1 - e^{-2C}) W_{S\alpha} W_S.x^T \right. \\
&\quad + (1 - e^{2C}) (W_{1S} - W_{2S})_\alpha (W_{1S} - W_{2S}).x^T + \hbar^2 P^2 (1 - \cosh 2C) x_\alpha^T \\
&\quad \left. - 4(\sinh 2C) \frac{W_{1S}.x^T W_{2S}.x^T}{x^{T2}} x_\alpha^T + 8\dot{C} W_{1S}.x^T W_{2S}.x^T x_\alpha^T \right\} .
\end{aligned} \tag{B4}$$

Here A , B , C are functions of x^{T2} (and eventually of P^2).

The operator w can be decomposed along projectors, w_0 and w_1 , on spin 0 and spin 1 states, respectively, which also satisfy with w_{12} the following relations :

$$\begin{aligned}
w &= 3w_0 - w_1 , \quad w_0 = \frac{1}{4}(1+w) , \quad w_1 = \frac{1}{4}(3-w) , \\
w(1-w_{12}) &= -(1-w_{12}) , \quad w(1+w_{12}) = 2w+1-w_{12} , \\
w_0 \frac{1}{2}(1-w_{12}) &= 0 , \quad w_0 \frac{1}{2}(1+w_{12}) = w_0 , \\
w_1 \frac{1}{2}(1-w_{12}) &= \frac{1}{2}(1-w_{12}) , \quad w_1 \frac{1}{2}(1+w_{12}) = \frac{1}{2}(1+w_{12}) - w_0 , \\
e^{-w_{12}C} &= \frac{1}{2}(1-w_{12}) e^C + \frac{1}{2}(1+w_{12}) e^{-C} , \\
e^{A+wB} &= w_0 e^{A+3B} + w_1 e^{A-B} .
\end{aligned} \tag{B5}$$

We also list here several useful formulas (in the c.m. frame) involving the momentum, the spin and the orbital angular momentum operators [Eqs. (3.18)] :

$$[\mathbf{p}^2 , f] = - (4i\hbar f' \mathbf{x} \cdot \mathbf{p} + 6\hbar^2 f' + 4\hbar^2 \mathbf{x}^2 f'') , \quad (\text{B6a})$$

$$\begin{aligned} [\mathbf{p}^2 , S_{12} g] &= - S_{12} (4i\hbar g' \mathbf{x} \cdot \mathbf{p} + 6\hbar^2 g' + 4\hbar^2 \mathbf{x}^2 g'') \\ &+ \frac{1}{\mathbf{x}^2} (4S_{12} \mathbf{L} \cdot \mathbf{S} - 8\mathbf{s}_1 \cdot \mathbf{s}_2 + 6\hbar^2 S_{12}) g \end{aligned} \quad (\text{B6b})$$

(f and g are functions of \mathbf{x}^2 and eventually of P^2 and the prime operations are defined in Eqs. (3.21)),

$$\begin{aligned} (\mathbf{s}_1 \pm \mathbf{s}_2) \cdot \mathbf{x} (\mathbf{s}_1 \pm \mathbf{s}_2) \cdot \mathbf{p} &= \frac{1}{2} (1 \pm S_{12}) (\hbar^2 \mathbf{x} \cdot \mathbf{p} + i\hbar \mathbf{L} \cdot \mathbf{S}) , \\ (\mathbf{s}_1 \mp \mathbf{s}_2) \cdot \mathbf{x} (\mathbf{s}_1 \pm \mathbf{s}_2) \cdot \mathbf{p} &= \frac{i\hbar}{2} (1 \mp S_{12}) \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2) , \\ [S_{12} , \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2)]_+ &= 0 , \quad [S_{12} , [S_{12} , \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2)]]_+ = 0 , \\ w (1 + S_{12}) \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2) &= -(1 + S_{12}) \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2) , \\ w (1 - S_{12}) \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2) &= 3(1 - S_{12}) \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2) , \\ (1 \pm S_{12}) \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2) (1 \pm S_{12}) &= 0 , \\ (1 \pm S_{12}) \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2) (1 \mp S_{12}) &= 2\mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2) \pm [S_{12} , \mathbf{L} \cdot (\mathbf{s}_1 - \mathbf{s}_2)] , \\ [S_{12} , \mathbf{L} \cdot \mathbf{S}]_+ &= 2\mathbf{S}^2 - 3\hbar^2 - 3\hbar^2 S_{12} , \\ (1 + S_{12}) \mathbf{L} \cdot \mathbf{S} (1 + S_{12}) &= -2\hbar^2 (1 + S_{12}) , \\ (1 - S_{12}) \mathbf{L} \cdot \mathbf{S} (1 - S_{12}) &= 4\hbar^2 (1 + S_{12}) - 4\mathbf{S}^2 , \\ (1 \pm S_{12}) \mathbf{L} \cdot \mathbf{S} (1 \mp S_{12}) &= 2 (\mathbf{L} \cdot \mathbf{S} + \mathbf{S}^2 - \frac{(\mathbf{S} \cdot \mathbf{x})^2}{\hbar^2 \mathbf{x}^2}) \pm [S_{12} , \mathbf{L} \cdot \mathbf{S}] , \\ [S_{12} , [S_{12} , \mathbf{L} \cdot \mathbf{S}]]_+ &= 0 , \quad S_{12} [S_{12} , \mathbf{L} \cdot \mathbf{S}] = 2 (\mathbf{L} \cdot \mathbf{S} + \mathbf{S}^2 - \frac{(\mathbf{S} \cdot \mathbf{x})^2}{\mathbf{x}^2}) \end{aligned} \quad (\text{B7})$$

($[,]_+$ is the anticommutator),

$$4 \mathbf{s}_1 \cdot \mathbf{p} \mathbf{s}_2 \cdot \mathbf{p} = \hbar^2 S_{12} \mathbf{p}^2 + \frac{i\hbar}{\mathbf{x}^2} [S_{12} , \mathbf{L} \cdot \mathbf{S}] \mathbf{x} \cdot \mathbf{p} - \frac{1}{\mathbf{x}^2} [S_{12} , \mathbf{L} \cdot \mathbf{S}] \mathbf{L} \cdot \mathbf{S} . \quad (\text{B8})$$

The relation expressing ψ_4 in terms of ψ_3 is :

$$\begin{aligned}
\frac{1}{4}MP_L \psi_4 = & \left[\frac{1}{2}(1 + S_{12}) \frac{P^2}{4} e^{2(V_2 - V_1)} + \frac{1}{2}(1 - S_{12}) \frac{M^2}{4} e^{2(V_1 - V_2)} \right] \psi_3 \\
& - e^{-[4U_4 + 2(V_1 + V_2) + 2h]} \left\{ \frac{1}{2\hbar^2 \mathbf{x}^2} [S_{12}, \mathbf{L.S}] (-i\hbar \mathbf{x.p} + \mathbf{L.S}) \right. \\
& + \frac{i\hbar}{2\hbar^2 \mathbf{x}^2} (1 - e^{-2T_4}) [S_{12}, \mathbf{L.S}] \mathbf{x.p} \\
& - [2(V'_1 - V'_2 + V'_3 + U'_4) e^{-2T_4} - \frac{1}{2\mathbf{x}^2}(1 - e^{-2T_4})] (\mathbf{L.S} + \mathbf{S}^2 - \frac{(\mathbf{S.x})^2}{\mathbf{x}^2}) \\
& \left. + [T'_4 e^{-2T_4} + \frac{3}{4\mathbf{x}^2} (1 - e^{-2T_4})] [S_{12}, \mathbf{L.S}] \right\} \psi_3 .
\end{aligned} \tag{B9}$$

Notice that the terms in the curly brackets disappear in the sectors with quantum numbers $j = \ell$.

The relation expressing ψ_3 in terms of ψ_4 can be deduced from Eq. (B9) with the following substitutions : $\psi_3 \leftrightarrow \psi_4$, $M^2 \leftrightarrow P^2$, $V_1 \leftrightarrow V_2$ and a global change of sign in front of the curly brackets.

Equations (3.14), which express ψ_1 and ψ_2 in terms of ψ_3 , can be rewritten in more explicit forms when potential V is given by Eq. (2.20) :

$$\begin{aligned}
\psi_1 = & \frac{2}{\hbar P_L} \times e^{-2(U_4 + V_2 + h)} \left\{ (\mathbf{s}_1 - \mathbf{s}_2). \mathbf{p} - \frac{1}{\mathbf{x}^2} (1 - e^{-2T_4}) (\mathbf{s}_1 - \mathbf{s}_2). \mathbf{x} \mathbf{x.p} \right. \\
& + \frac{i\hbar}{2\mathbf{x}^2} (1 - e^{-2T_4}) \left(4 - \frac{2\mathbf{S}^2}{\hbar^2} \right) (\mathbf{s}_1 - \mathbf{s}_2). \mathbf{x} \\
& - 2i\hbar (V'_1 - V'_2 + V'_3 + U'_4 - T'_4 - \frac{2\mathbf{S}^2}{\hbar^2} U'_4) e^{-2T_4} (\mathbf{s}_1 - \mathbf{s}_2). \mathbf{x} \\
& - \frac{(m_1^2 - m_2^2)}{M^2} e^{-4V_1} \left[\mathbf{S.p} - \frac{1}{\mathbf{x}^2} (1 - e^{-2T_4}) \mathbf{S.x} \mathbf{x.p} \right. \\
& + \frac{i\hbar}{\mathbf{x}^2} (1 - e^{-2T_4}) \mathbf{S.x} \\
& \left. \left. + 2i\hbar (V'_1 - V'_2 + V'_3 + U'_4 + T'_4) e^{-2T_4} \mathbf{S.x} \right] \right\} \psi_3 ,
\end{aligned} \tag{B10a}$$

$$\begin{aligned}
\psi_2 = & \frac{2}{\hbar M} \times e^{-2(U_4 + V_1 + h)} \left\{ - \mathbf{S} \cdot \mathbf{p} + \frac{1}{\mathbf{x}^2} (1 - e^{-2T_4}) \mathbf{S} \cdot \mathbf{x} \mathbf{x} \cdot \mathbf{p} \right. \\
& - \frac{i\hbar}{\mathbf{x}^2} (1 - e^{-2T_4}) \mathbf{S} \cdot \mathbf{x} \\
& - 2i\hbar (V'_1 - V'_2 + V'_3 + U'_4 + T'_4) e^{-2T_4} \mathbf{S} \cdot \mathbf{x} \\
& + \frac{(m_1^2 - m_2^2)}{P^2} e^{-4V_2} \left[(\mathbf{s}_1 - \mathbf{s}_2) \cdot \mathbf{p} - \frac{1}{\mathbf{x}^2} (1 - e^{-2T_4}) (\mathbf{s}_1 - \mathbf{s}_2) \cdot \mathbf{x} \mathbf{x} \cdot \mathbf{p} \right. \\
& + \frac{i\hbar}{2\mathbf{x}^2} (1 - e^{-2T_4}) \left(4 - \frac{2\mathbf{S}^2}{\hbar^2} \right) (\mathbf{s}_1 - \mathbf{s}_2) \cdot \mathbf{x} \\
& \left. \left. - 2i\hbar (V'_1 - V'_2 + V'_3 + U'_4 - T'_4 - \frac{2\mathbf{S}^2}{\hbar^2} U'_4) e^{-2T_4} (\mathbf{s}_1 - \mathbf{s}_2) \cdot \mathbf{x} \right] \right\} \psi_3 .
\end{aligned}
\tag{B10b}$$

APPENDIX C: CANCELLATION OF HIGH ORDER LADDER DIAGRAMS IN THE INFINITE MASS LIMIT

We show in this appendix that in the infinite mass limit ($m_2 \rightarrow \infty$) the high order ladder and crossed ladder diagrams are cancelled by the “constraint diagrams”, so that the dominant contribution comes from the one-particle exchange diagrams only.

We shall ignore the contributions of vertex corrections, which yield nonlocal terms in the potential, even in the one-particle problem. For simplicity, internal fermion propagators will be considered as free, but this assumption does not restrict the generality of the result. Vacuum polarization may be included in the exchanged particle propagator. For definiteness, we consider the case of QED (with coupling constants e_1 and e_2 for fermions 1 and 2, respectively), but the derivation can be repeated with other types of interaction as well. Mass of the exchanged particle is irrelevant.

In the approximation of ladder and crossed ladder diagrams, the off-mass shell (amputated) scattering amplitude can be decomposed as :

$$T = \sum_{n=1}^{\infty} T_n , \quad (C1)$$

where T_n is the partial amplitude corresponding to n exchanged photons. A similar decomposition also holds for the constrained amplitude \tilde{T} (2.10).

The iteration of Eq. (2.10) yields for the potential the expression :

$$\tilde{V} = \tilde{T} \sum_{p=0}^{\infty} (G_0 \tilde{T})^p \equiv \sum_{n=0}^{\infty} \tilde{V}_n , \quad (C2)$$

where \tilde{V}_n is that part of the potential which comes from the contributions of n exchanged photons. The first two terms of \tilde{V} are :

$$\tilde{V}_1 = \tilde{T}_1 , \quad (C3)$$

$$\tilde{V}_2 = \tilde{T}_2 + \tilde{T}_1 G_0 \tilde{T}_1 . \quad (C4)$$

The corresponding Feynman diagrams are represented in Fig. 1. While \tilde{T}_2 contains the two

usual Feynman diagrams of two exchanged photons, the last term in Eq. (C4) yields the “constraint diagram” (third diagram in Fig. 1b).

The expression of \tilde{V}_1 was given in Appendix A (notice that to this order the potentials \tilde{V} and V [Eq. (2.17)] coincide); in the limit $m_2 \rightarrow \infty$, \tilde{V}_1 behaves as $O(1/m_2)$. We now concentrate on the behavior of \tilde{V}_2 .

In \tilde{T} , the external momenta are submitted to the constraint (2.11) and, therefore, satisfy the relations :

$$\begin{aligned} p_1 + p_2 &= p'_1 + p'_2 = P , \\ p_1^2 - p_2^2 &= p'^2_1 - p'^2_2 = m_1^2 - m_2^2 , \quad p'_{1L} = p_{1L} , \quad p'_{2L} = p_{2L} . \end{aligned} \quad (\text{C5})$$

In the limit $m_2 \rightarrow \infty$, one has :

$$\begin{aligned} p_{20} &= m_2 + O\left(\frac{1}{m_2}\right) , \quad P_0 = m_2 + p_{10} + O\left(\frac{1}{m_2}\right) , \\ p_{10} - p'_{10} &= p'_{20} - p_{20} = O\left(\frac{|\mathbf{P}|}{m_2}\right) . \end{aligned} \quad (\text{C6})$$

The expression of \tilde{T}_2 is :

$$\begin{aligned} \tilde{T}_2 &= (-i)^4 (e_1 e_2)^2 \left(\frac{i}{2P_L} \right) \int \frac{d^4 k_1}{(2\pi)^4} D_{\mu_1 \nu_1}(k_1) D_{\mu_2 \nu_2}(p_1 - p'_1 - k_1) \\ &\quad \times \gamma_{1\mu_1} S_1(p_1 - k_1) \gamma_{1\mu_2} [\gamma_{2\nu_1} S_2(-(p_2 + k_1)) \gamma_{2\nu_2} + \gamma_{2\nu_2} S_2(-(p'_2 - k_1)) \gamma_{2\nu_1}] , \end{aligned} \quad (\text{C7})$$

where the photon propagator is considered in an arbitrary covariant gauge [Eq. (A5)].

In the limit $m_2 \rightarrow \infty$ one has [17,18] :

$$\begin{aligned} &[\gamma_{2\nu_1} S_2(-(p_2 + k_1)) \gamma_{2\nu_2} + \gamma_{2\nu_2} S_2(-(p'_2 - k_1)) \gamma_{2\nu_1}] \\ &= [\gamma_{2\nu_1} \frac{i}{-\gamma_2 \cdot (p_2 + k_1) - m_2 + i\epsilon} \gamma_{2\nu_2} + \gamma_{2\nu_2} \frac{i}{-\gamma_2 \cdot (p'_2 - k_1) - m_2 + i\epsilon} \gamma_{2\nu_1}] \\ &= (-1)^2 \delta_{\nu_1 0} \delta_{\nu_2 0} 2\pi \delta(k_{10}) + O\left(\frac{1}{m_2}\right) , \end{aligned} \quad (\text{C8})$$

where the eigenvalue -1 has been used for the matrix γ_{20} (antifermion). (Because of the factor $\delta(k_{10})$ and Eqs. (C6), the gauge part of the photon propagator disappears in Eq. (C7) at leading order.)

On the other hand, in the iteration term $\tilde{T}_1 G_0 \tilde{T}_1$ [Eq. (C4)] the integration concerning G_0 is three-dimensional after the use of the constraint (2.11) with the fermion momenta $(p_1 + k_1)$ and $(p_2 - k_1)$. This yields the condition $k_{1L} = 0$. Hence, one obtains :

$$\begin{aligned} \tilde{T}_1 G_0 \tilde{T}_1 &= (-i)^4 (e_1 e_2)^2 \left(\frac{i}{2P_L} \right)^2 \int \frac{d^4 k_1}{(2\pi)^4} 2\pi \delta(k_{1L}) D_{\mu_1 \nu_1}(k_1) D_{\mu_2 \nu_2}(p_1 - p'_1 - k_1) \\ &\quad \times \gamma_{1\mu_1} S_1(p_1 - k_1) \gamma_{1\mu_2} \gamma_{2\nu_1} S_2(-(p_2 + k_1)) \gamma_{2\nu_2} H_0(p_2 + k_1) . \end{aligned} \quad (C9)$$

Notice that because of the constraint (2.11) $H_0(p_1 - k_1) = H_0(p_2 + k_1)$ [Eq. (2.13)]. One also has :

$$S_2(-(p_2 + k_1)) H_0(p_2 + k_1) = i(-\gamma_2 \cdot (p_2 + k_1) + m_2) . \quad (C10)$$

In the limit $m_2 \rightarrow \infty$, this term, multiplied by the matrices $\gamma_{2\nu_1}$ and $\gamma_{2\nu_2}$, behaves as $(-1)^2 \delta_{\nu_{10}} \delta_{\nu_{20}} (i2m_2)$ and the factor $2m_2$ cancels one of the factors $2P_L$ of the denominator of the right-hand side of Eq. (C9). Furthermore, in the same limit $\delta(k_{1L}) \rightarrow \delta(k_{10})$ and, because of the additional factor i^2 , the iteration term $\tilde{T}_1 G_0 \tilde{T}_1$ cancels, at leading order, the amplitude term \tilde{T}_2 in Eq. (C4). Hence, \tilde{V}_2 behaves, when $m_2 \rightarrow \infty$, as $O(1/m_2^2)$.

The above calculations can be repeated at higher orders of perturbation theory. For the diagrams with n exchanged photons, \tilde{V}_n is the sum of the partial amplitude \tilde{T}_n and the iteration terms coming from $\tilde{V} G_0 \tilde{T}$; it can be written in the form :

$$\tilde{V}_n = \tilde{T}_n + \sum_{p=1}^{n-1} \sum_{r_1 + \dots + r_{p+1} = n} \tilde{T}_{r_1} G_0 \tilde{T}_{r_2} G_0 \dots \tilde{T}_{r_p} G_0 \tilde{T}_{r_{p+1}} , \quad (C11)$$

where, in the generic term of the sum, the constraint factor G_0 appears p times. A typical diagram, where $p = 2$, is shown in Fig. 2.

Putting aside all factors which do not play an essential role, an amplitude \tilde{T}_r , that contains all ladder and crossed ladder type Feynman diagrams with r exchanged photons, behaves in the limit $m_2 \rightarrow \infty$ as [17,18] :

$$\tilde{T}_r \sim (-i)^{2r} (-e_1 e_2)^r \left(\frac{i}{2P_L} \right) (2\pi)^{r-1} \delta(k_{10}) \dots \delta(k_{r-10}) , \quad (C12)$$

while the occurrence of a constraint factor G_0 just after \tilde{T}_r provides the contribution :

$$G_0 \sim (i2m_2) 2\pi\delta(k_{r0}) . \quad (\text{C13})$$

It can be checked , using these results, that a term of the iteration sum in Eq. (C11), containing p factors G_0 , behaves as :

$$\tilde{T}_{r_1} G_0 \tilde{T}_{r_2} G_0 \cdots \tilde{T}_{r_p} G_0 \tilde{T}_{r_{p+1}} \sim (-1)^p \tilde{T}_n . \quad (\text{C14})$$

Furthermore, the p factors G_0 may appear in such a term in $\binom{n-1}{p}$ different configurations. The behavior of \tilde{V}_n is then :

$$\lim_{m_2 \rightarrow \infty} m_2 \tilde{V}_n = \lim_{m_2 \rightarrow \infty} m_2 \tilde{T}_n \left[1 + \sum_{p=1}^{n-1} (-1)^p \binom{n-1}{p} \right] = 0 , \quad n \geq 2 . \quad (\text{C15})$$

Therefore, \tilde{V}_n behaves as $O(1/m_2^2)$ and the dominant contribution to \tilde{V} [Eq. (C2)] comes solely from the one-particle exchange diagram, which behaves as $O(1/m_2)$. A similar result is also obtained in other approaches, based on a three-dimensional formulation of the two-body bound state problem [13,22].

REFERENCES

- [1] *Constraint's Theory and Relativistic Dynamics*, edited by G. Longhi and L. Lusanna (World Scientific, Singapore, 1987), and references therein.
- [2] H.W. Crater and P. Van Alstine, Ann. Phys. (N.Y.) **148**, 57 (1983); Phys. Rev. D **36**, 3007 (1987).
- [3] H. Sazdjian, Phys. Rev. D **33**, 3401 (1986); J. Math. Phys. **29**, 1620 (1988).
- [4] H.W. Crater, R.L. Becker, C.Y. Wong and P. Van Alstine, Phys. Rev. D **46**, 5117 (1992).
- [5] E.E. Salpeter and H.A. Bethe, Phys. Rev. **84**, 1232 (1951);
M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).
- [6] N. Nakanishi, Suppl. Prog. Theor. Phys. **43**, 1 (1969); see also *Bibliography of the Bethe-Salpeter equation*, prepared by M.-T. Noda, N. Nakanishi and N. Setô, Suppl. Prog. Theor. Phys. **95**, 78 (1988).
- [7] H. Sazdjian, in *Extended Objects and Bound Systems*, proceedings of the Karuizawa International Symposium, 1992, edited by O. Hara, S. Ishida and S. Naka (World Scientific, Singapore, 1992), p. 117; J. Math. Phys. **28**, 2618 (1987).
- [8] A.A. Logunov and A.N. Tavkhelidze, Nuovo Cimento **29**, 380 (1963);
A.A. Logunov, A.N. Tavkhelidze, I.T. Todorov and O.A. Khrustalev, *ibid.* **30**, 134 (1963).
- [9] R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).
- [10] F. Gross, Phys. Rev. **186**, 1448 (1969); Phys. Rev. C **26**, 2203 (1982); *ibid.* 2226.
- [11] M.H. Partovi and E.L. Lomon, Phys. Rev. D **2**, 1999 (1970).
- [12] R.N. Faustov, Teor. Mat. Fiz. **3**, 240 (1970) [Theor. Math. Phys. **3**, 478 (1970)].

- [13] C. Fronsdal and R.W. Huff, Phys. Rev. D **3**, 933 (1971).
- [14] I.T. Todorov, Phys. Rev. D **3**, 2351 (1971); in *Properties of Fundamental Interactions*, edited by A. Zichichi (Editrice Compositori, Bologna, 1973), Vol. 9, Part C, p. 951.
- [15] G.P. Lepage, Phys. Rev. A **16**, 863 (1977);
W.E. Caswell and G.P. Lepage, *ibid.* **18**, 810 (1978); **20**, 36 (1979).
- [16] H.W. Crater and P. Van Alstine, J. Math. Phys. **31**, 1998 (1990).
- [17] S.J. Brodsky, in *Brandeis Lectures 1969*, edited by M. Chrétien and E. Lipworth (Gordon and Breach, New York, 1971), p. 95, Secs. 3 and 4.
- [18] A.R. Neghabian and W. Glöckle, Can. J. Phys. **61**, 85 (1983).
- [19] G.T. Bodwin and D.R. Yennie, Phys. Reports **43**, 267 (1978).
- [20] G. Feldman, T. Fulton and D.L. Heckathorn, Nucl. Phys. **B167**, 364 (1980); **B174**, 89 (1980).
- [21] L.D. Landau and I.M. Khalatnikov, Sov. Phys. JETP **2**, 69 (1956);
B. Zumino, J. Math. Phys. **1**, 1 (1959);
K. Johnson and B. Zumino, Phys. Rev. Lett. **3**, 351 (1959);
R. Barbieri, M. Ciafaloni and P. Menotti, Nuovo Cimento **55A**, 701 (1968).
- [22] V.A. Rizov, I.T. Todorov and B.L. Aneva, Nucl. Phys. **B98**, 447 (1975).
- [23] H.A. Bethe and E.E. Salpeter, *Quantum Mechanics of One-and Two-Electron Atoms* (Springer Verlag, Berlin, 1957), p. 193;
V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii, *Relativistic Quantum Theory* (Pergamon Press, Oxford, 1971), Vol. 4, Part 1, p. 283.
- [24] M. Bawin, J. Cugnon and H. Sazdjian, preprint IPNO-TH 93-41.
- [25] M.G. Olsson and K.J. Miller, Phys. Rev. D **28**, 674 (1983).

- [26] G.C. Bhatt, H. Grotch and Xingguo Zhang, J. Phys. G **17**, 231 (1991).
- [27] A.B. Henriques, B.H. Keller and R.G. Moorhouse, Phys. Lett. **64B**, 85 (1976);
H.J. Schnitzer, Phys. Lett. **76B**, 461 (1978); Phys. Rev. D **18**, 3482 (1978);
T. Barnes and G.I. Ghandour, Phys. Lett. **118B**, 411 (1982);
R. McClary and N. Byers, Phys. Rev. D **28**, 1692 (1983).
- [28] A. Barchielli, E. Montaldi and G.M. Prosperi, Nucl. Phys. **B296**, 625 (1988); **B303**,
752 (E) (1988);
N. Brambilla and G.M. Prosperi, Phys. Lett. B **236**, 69 (1990);
A. Barchielli, N. Brambilla and G.M. Prosperi, Nuovo Cimento **103A**, 59 (1990).
- [29] K. Wilson, Phys. Rev. D **10**, 2445 (1974);
T. Appelquist, M. Dine and I.J. Muzinich, *ibid.* **17**, 2074 (1978);
L.S. Brown and W.I. Weisberger, *ibid.* **20**, 3239 (1979).
- [30] E. Eichten and F. Feinberg, Phys. Rev. D **23**, 2724 (1981).
- [31] D. Gromes, Z. Phys. C **26**, 401 (1984); **22**, 265 (1984).
- [32] D. LaCourse and M.G. Olsson, Phys. Rev. D **39**, 2751 (1989);
C. Olson, M.G. Olsson and K. Williams, *ibid.* **45**, 4307 (1992);
M.G. Olsson and K. Williams, *ibid.* **48**, 417 (1993).

FIGURES

FIG. 1. The diagrams contributing to \tilde{V}_1 , (a), and \tilde{V}_2 , (b) [Eqs. (C2)-(C4)]. The third diagram in (b) is the “constraint diagram”.

FIG. 2. A typical “constraint diagram” of high order [Eq. (C11)], where the constraint factor G_0 [Eq. (2.12)] appears twice (boxes with a cross). The boxes without a cross represent ordinary ladder and crossed ladder type Feynman diagrams (contributing to the partial amplitudes \tilde{T}_r in Eq. (C11)).

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